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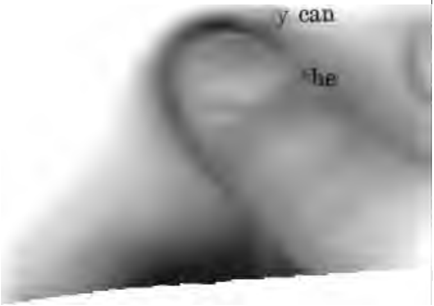

196 – 21 Let x = no of days worked together, frac each one can do in one day $\frac{1}{x}$ = fraction working together in one day $\frac{1}{4}+\frac{1}{5}=$ [illegible]

196 – 21 x = no days of together.

can do 1/4 [illegible] 1/5

1 day | 1 | x days

1/4 | $\frac{x}{4}+\frac{x}{5}$

$\frac{1}{4}+$ [illegible] $=1$

$\frac{x}{4}+\frac{x}{5}=3/4$

$5x+4x=15$

$9x=15$

$x=1\frac{2}{3}$

196 – 26

102 – 1

$\frac{1}{x}+\frac{1}{y}+\frac{1}{3}=$ [illegible]

[illegible] $\frac{3}{4}$

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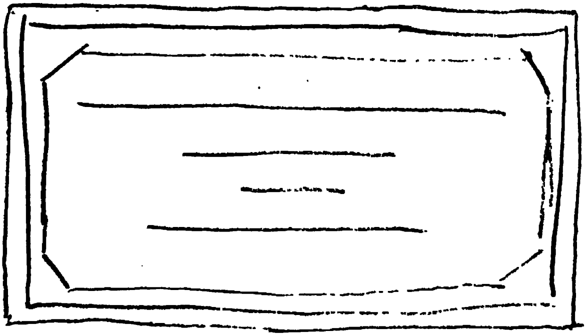

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PREFACE

The present revision of the "Complete School Algebra" bears substantially the same relation to the revised "First Course" and "Second Course" that the first edition bore to the original texts. In the revision of each of these texts material for which there is no strong demand from teachers has been omitted, and the entire work has been rewritten in the interest of greater simplicity and directness of appeal. The collections of exercises and problems are for the most part new and contain a larger proportion of easy exercises with simple results than the first edition.

A striking feature of the revision is the inclusion of a large number of oral exercises in connection with the introduction of each new idea or operation. It is the object of these exercises to present the new concept in complete isolation from any complication of notation or technique so that the student becomes familiar with its content and bearing before he is asked to make use of it in written work. These oral exercises may well be taken up when the advance lesson is assigned, so that the pupil may be certain that he understands the idea involved in the new work before he leaves his instructor.

Another feature scarcely less important is the character and position of the examples and hints. The aim has been to help the student at the exact point where he needs it and to avoid the insertion of lengthy and difficult solutions before they can be completely understood.

The definitions and axioms have been expressed in the simplest language which is consistent with scientific accuracy. Many definitions which are usually found in elementary texts but which do not contribute to the clearness of the subject are omitted.

The first presentation of the subject of graphs has been limited to the study of the straight line and a few exercises of a commercial or scientific character. These exercises not only have a very definite human interest apart from their mathematical value but also serve to familiarize the student with the kind of graphs he will meet in his ordinary reading.

In the chapter on Functions and their Graphs the attention has been centered on the graphical representation of a function, on the graphical solution of an equation in one unknown, and on that of a system in two unknowns. A few carefully selected statistical problems, each showing some striking feature when treated graphically, have been included. The whole is designed to secure, with as little labor as possible, the maximum of illumination and interest.

In the chapters devoted to a review of first-year algebra the fact was borne constantly in mind that the material might be handled by students who had not pursued the study of algebra during the preceding year. It was consequently thought desirable to have work in equations as early as practicable. The subject of fractions, the topic usually most in need of review, has received full and careful treatment. Linear systems have been presented without the use of determinants. The needs of classes, even under almost identical conditions, differ widely, one class requiring more review on a certain topic than does another. Consequently the review material has been expanded so as to afford ample work for any class. It is not intended, however, that all the exercises and problems should be solved by any one student.

The authors are under obligations to many teachers from all parts of the country for helpful criticisms which have been of material assistance in planning and carrying forward this revision.

CONTENTS

CHAPTER PAGE

I. INTRODUCTION (Sects. 1-11) 1

II. POSITIVE AND NEGATIVE NUMBERS (Sects. 12-19) 18

III. ADDITION (Sects. 20-25) 33

IV. SIMPLE EQUATIONS (Sects. 26-28) 39

V. SUBTRACTION (Sects. 29-30) 49

VI. IDENTITIES AND EQUATIONS OF CONDITION (Sects. 31-34) 54

VII. PARENTHESES (Sects. 35-36) 64

VIII. MULTIPLICATION (Sects. 37-44) 70

IX. PARENTHESES IN EQUATIONS (Sects. 45-46) . . 79

X. DIVISION (Sects. 47-49) 87

XI. EQUATIONS AND PROBLEMS (Sects. 50-51) . . . 95

XII. IMPORTANT SPECIAL PRODUCTS (Sects. 52-55) . . 105

XIII. FACTORING (Sects. 56-66) 113

XIV. SOLUTION OF EQUATIONS BY FACTORING (Sects. 67-70) 137

XV. FRACTIONS (Sects. 71-81) 148

XVI. EQUATIONS CONTAINING FRACTIONS (Sects. 82-88) 175

XVII. GRAPHICAL REPRESENTATION (Sects. 89-94) . . 200

XVIII. LINEAR SYSTEMS (Sects. 95-100) 217

XIX. SQUARE ROOT (Sects. 101-102) 240

XX. RADICALS (Sects. 103-114) 250

XXI. QUADRATIC EQUATIONS (Sects. 115-117) 270

CHAPTER PAGE

XXII. Review of First Year Algebra (Sects. 118–156) 282

XXIII. Exponents (Sects. 157–159) 329

XXIV. Square Root and Radicals (Sects. 160–176) . . 338

XXV. Functions and their Graphs (Sects. 177–186) . 359

XXVI. Quadratic Equations (Sects. 187–189) 369

XXVII. Irrational Equations (Sect. 190) 381

XXVIII. Imaginaries (Sects. 191–196) 386

XXIX. Theory of Quadratic Equations (Sects. 197–201) 396

XXX. Graphs of Quadratic Equations in Two Variables (Sects. 202–204) 405

XXXI. Systems Solvable by Quadratics (Sects. 205–210) 416

XXXII. Progressions (Sects. 211–220) 432

XXXIII. The Binomial Theorem (Sects. 221–225) . . . 455

XXXIV. Ratio, Proportion, and Variation (Sects. 226–236) 463

XXXV. Logarithms (Sects. 237–249) 480

INDEX 503

LIST OF PORTRAITS

John Wallis 48

Sir William Rowan Hamilton 70

Sir Isaac Newton 100

René Descartes 210

François Vieta 268

Sir John Napier 462

COMPLETE SCHOOL ALGEBRA

CHAPTER I

INTRODUCTION

1. The numbers and symbols of arithmetic. The simple operation of counting employs the numbers we call integers. To represent these integers and the other numbers with which it deals, arithmetic uses the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Operations on the numbers of arithmetic are indicated by the symbols $+$, $-$, $\times$, and $\div$. The operation of division applied to integers gives rise to fractions. With these two kinds of numbers, integers and fractions, the student's work in arithmetic is mainly carried on.

2. Symbols of algebra. Symbols are employed far more extensively in algebra than in arithmetic, and many new ideas arise in connection with their meaning and their use. Some symbols represent numbers, others indicate operations upon them, others represent relations between them, and still others represent kinds of numbers with which arithmetic does not deal. Letters as well as arabic numerals are used to represent numbers. The following symbols of operation, $+$, $-$, $\times$, and $\div$, have the same meaning as in arithmetic. The sign of multiplication is usually replaced by a dot or omitted.

For example, $3 \times a$ is written $3 \cdot a$, or $3a$, and $2 \times a \times b$ is written $2ab$. Also $a \div b$ is often written $\frac{a}{b}$

The sign = is read *equals*, or *is equal to*. As the need for them arises, other symbols will be introduced.

3. The use of letters to represent numbers. The use of the letters of the alphabet to represent numbers is the most striking difference between arithmetic and algebra.

In arithmetic we speak thus: If the sides of a triangle are 6, 7, and 9 inches respectively, its perimeter is $6+7+9$, or 22 inches. The corresponding statement in algebra is: If the sides of a triangle are a, b, and c inches respectively and its perimeter is p inches, then $p=a+b+c$. Here the second statement is true for every triangle, while the first is not true for every triangle.

Similarly: If a rectangle is 8 inches by 12 inches, its perimeter is $8+12+8+12$, or 40 inches. And if a rec-tangle is l inches long and w inches wide and if p denotes its perimeter in inches, then $p=l+w+l+w$, or $2l+2w$ Here, again, the arithmetical statement is particular and applies to one rectangle only, while the algebraic statement is general; that is, it is true for all rectangles.

The gain in power which the general symbolic languag of algebra affords over the particular numerical language of arithmetic constitutes one of the most important advan tages of the algebraic method. As the student progresses h will meet with many illustrations of this feature of algebra

The purpose of the following exercises is to familiarize the student with the use of letters in the place of numbers

ORAL EXERCISES

1. [illegible] has $5a$ when a is 3? when a is 5

[illegible] has $6a+2b$ when a is 2 and b is 4
[illegible] seconds if h and m stand for th
[illegible] in a minute respectively.

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28. The side of a square is s inches long. What represents its perimeter? its area?

29. The base of a rectangle is 12 feet, and its altitude is 4 feet. What is its perimeter? its area?

30. If b represents the number of feet in the base of a rectangle and a the number of feet in its altitude, what is its perimeter? its area?

31. A rectangle is twice as long as it is wide. Let w represent the number of inches in its width. Then express, in terms of w, (*a*) the length; (*b*) the perimeter; (*c*) the area.

32. A man is three times as old as his son. If s denotes the number of years in the son's age, what will represent the father's age?

33. A father is 28 years older than his son. If s represents the son's age in years, what will represent the father's age?

34. A rectangle is 24 inches longer than it is wide. Let b represent the width in feet. Then represent the length and the perimeter in terms of b and numbers.

35. A rectangle is 16 feet narrower than it is long. If w represents the width in feet, what will conveniently represent the length? the perimeter?

36. A rectangle is 4 feet longer than twice its width. Express the width, the length, and the perimeter in terms of a letter, or a letter and numbers.

Origin of symbols. Many of the symbols that are in common use in algebra at the present time have histories which not only are interesting in themselves, but which also serve to indicate the slow and uncertain development of the subject. It is often found that symbols which seem without meaning represent some abbreviation or suggestion long since forgotten, and that operations and methods which we find hard to master have sometimes required hundreds of years to perfect.

In the early centuries there were practically no algebraic symbols in common use; one wrote out in full the words *plus*, *minus*, *equals*, and the like. But in the sixteenth century several Italian mathematicians used the initial letters $\bar{p}$ and $\bar{m}$ for + and −. Some think that our modern symbol − came into use through writing the initial *m* so rapidly that the curves of the letter gradually flattened out, leaving finally a straight line. The symbol + may have originated similarly in the rapid writing of the letter *p*. But in the opinion of others these symbols were first used in the German warehouses of the fifteenth century to mark the weights of boxes of goods. If a lot of boxes, each supposed to weigh 100 pounds, came to the warehouse, the weight would be checked, and if a certain box exceeded the standard weight by 5 pounds, it was marked $100 + 5$; if it lacked 5 pounds, it was marked $100 - 5$. Though the first book to use these symbols was published in 1489, it was not until about 1630 that they could be said to be in common use.

Both of the symbols for multiplication given in the text were first used about 1630. The cross was used by two Englishmen, Oughtred and Harriot, and was probably an adaptation of the letter *x*, which is found some years earlier. The dot is first found in the writings of the Frenchman Descartes. It is interesting to note that Harriot was sent to America in 1585 by Sir Walter Raleigh, and returned to England with a report of observations. He made the first survey of Virginia and North Carolina, and constructed maps of those regions.

It is strange that the line was used to denote division long before any of the other symbols here mentioned were in use. This is, in fact, one of the oldest signs of operation that we have. The Arabs, as early as 1000 A.D., used both $\frac{a}{b}$ and a/b to denote the quotient of *a* and *b*. The symbol ÷ did not occur until about 1630.

Equality has been denoted in a variety of ways. The word *equals* was usually written out in full until about the year 1600, though the two sides of an equation were written one over the other by the Hindus as early as the twelfth century. The modern sign = was probably introduced by the Englishman Recorde, in 1557, because, he says, "Noe. 2. thynges can be moare equalle" than two parallel lines. This symbol was not generally accepted at first, and in its place the symbols ||, ∝, and ∞ are frequently met during the next fifty years.

4. The usefulness of symbols. Symbols enable one to abbreviate ordinary language in the solution of problems.

For example: Three times a certain number is equal to 20 diminished by 5. What is the number?

If n represents the number, the preceding statement and question can be written in symbols, thus:

$$3n = 20 - 5.$$
$$n = ?$$

The symbolic statement $3n = 20 - 5$ is called an *equation* and n the *unknown number.*

If $3n = 20 - 5,$

then $3n = 15,$

and $n = 5.$

While the preceding example is very simple, it illustrates the algebraic method of stating and solving the problem. The method is brief and direct, and its advantages will become more apparent as the student progresses.

ORAL EXERCISES

Find the numerical value of x in the following equations:

1. $3x = 18.$

2. $4x = 28.$

3. $5x = 30.$

4. $7x = 42.$

5. $4x = 12 + 8.$

6. $3x = 4 + 11.$

7. $6x = 17 + 13.$

8. $4x + 3x = 35.$

9. $6x + 2x = 32.$

10. $5x + 4x = 45.$

11. $4x + 3x = 56.$

12. $7x + 2x = 15 + 3.$

13. $9x - 3x = 18 + 12.$

14. $4x - x = 15 - 6.$

15. $5x + 4x - 2x = 10 + 4.$

16. $4x + 3x - x = 33 - 3.$

17. $6x - x + 3x = 42 + 6.$

18. If one number is represented by x, what will represent a number three times as great?

19. James had $3x$ cents. His brother had four times as many. Represent the number of cents the brother had.

20. Paul's weight is $2x$ pounds, and his father weighs three times as much. What will represent the father's weight? the weight of the two together?

21. The area of a circle is $6y$. Represent the area of a circle three times as large.

22. One number is twice a second, and the second is four times the third. If x represents the third, what will represent the second? the first?

23. One newsboy has three times as many papers as a second, and the two together have as many as a third. Represent in terms of x the number of papers each has.

EXAMPLE

The sum of two numbers is 112. The greater is three times the less. What are the numbers?

Solution. By the conditions of the problem,

$$\text{greater number} + \text{less number} = 112. \qquad (1)$$

Let $l =$ the less number.

Then $3l =$ the greater number.

Substituting these symbols in (1), we have

$$3l + l = 112.$$

Collecting, $4l = 112.$

Whence $l = \frac{112}{4} = 28,$

and $3l = 3 \times 28 = 84.$

Therefore the greater number is 84 and the less 28.

We may verify the result by substituting 84 and 28 in the problem.

Thus $84 + 28 = 112,$

and $84 = 3 \cdot 28.$

PROBLEMS

1. The sum of two numbers is 120. The greater is four times the less. Find each.

2. A certain number plus seven times itself equals 216. Find the number.

3. One number is eight times another. Their sum is 72. Find each.

4. The first of three numbers is twice the third, and the second is four times the third. The sum of the three numbers is 252. Find the numbers.

HINT. Let $x =$ the third number. Then $2x =$ the first, and $4x =$ the second.

5. The sum of three numbers is 117. The second is twice the first, and the third is three times the second. Find each.

6. There are three numbers whose sum is 192. The first is twice the second, and the third equals the sum of the other two. Find the numbers.

7. The sum of three numbers is 312. The second is five times the first, and the third is four times the second. Find the numbers.

8. The sum of three numbers is 208. The second is three times the first, and the third is the sum of the other two. Find the numbers.

9. A man is three times as old as his son. The sum of their ages is 44 years. Find the age of each.

10. The perimeter of a certain square is 160 feet. Find the length of each side.

11. The perimeter of a certain rectangle is 216 feet. It is three times as long as it is wide. Find its dimensions.

12. The perimeter of the rectangle formed by placing two equal squares side by side is 258 inches. Find the side and the perimeter of each square.

5. Literal notation. In algebra numbers are represented by one or more arabic numerals, or by letters, or by both combined.

Thus 3, 25, a, $2b$, $4xy$, and $2x+3$ are algebraic symbols for numbers.

Precisely what numbers $4xy$ and $2x+3$ represent is not known until the numbers for which x and y stand are known. In one problem these letters may have values quite different from those they have in another. To devise methods of determining these values in the various problems which arise is the principal aim of algebra.

6. Factors. A **factor** of a product is any one of the numbers which multiplied together form the product.

Thus $3ab$ means 3 times a times b. Here 3, a, and b are each factors of $3ab$. Similarly, the expression $4(a+b)$ means 4 times the sum of a and b. Here 4 and $a+b$ are factors of $4(a+b)$.

ORAL EXERCISES

1. Name the factors in $3 \cdot 4 \cdot 6$, $2xy$, $3abx$, $4abc$.

In Exercises 2–5, replace a by 3 and b by 4 and find the value of the resulting expression.

2. ab. **3.** $3ab$. **4.** $2ab$. **5.** $5ab$.

7. Exponents. An **exponent** is an integer written at the right of and above another number to show how many times the latter is to be taken as a factor.

(Later this definition will be modified so as to include fractions and other numbers as exponents.)

Thus $3^2 = 3 \cdot 3$; $5^3 = 5 \cdot 5 \cdot 5$. Also $a^4 = a \cdot a \cdot a \cdot a$, and $4xy^3 = 4 \cdot x \cdot y \cdot y \cdot y$. In a^b, b is the exponent of a. If a is 4 and b is 3, $a^b = 4^3 = 4 \cdot 4 \cdot 4$. The exponent 1 is usually not written.

ORAL EXERCISES

1. What are the exponents in $2ac^3$? $3a^2c$? $5a^4x^2$?

2. What is meant by x^2? a^3? b^4? b^5?

3. $4^2 = ?$ **4.** $5^2 = ?$ **5.** $2^3 = ?$ **6.** $3 \cdot 5^2 = ?$

In Exercises 7–14, replace a by 3 and b by 2 and find the value of the result.

7. a^2. **9.** a^3. **11.** a^2b. **13.** $2a^2$.
8. $a^2 + b^2$. **10.** $a^3 + b^3$. **12.** a^2b^3. **14.** $5a^3b^2$.

8. Coefficients. If a number is the product of two factors, either of these factors is called the **coefficient** of the other in that product.

Thus in $4x^2y$, 4 is the coefficient of x^2y, y is the coefficient of $4x^2$, and $4y$ is the coefficient of x^2. The numerical coefficient 1 is usually omitted. If a numerical coefficient other than 1 occurs, it is usually written first. For instance, we write $5x$, not $x5$.

The following examples illustrate the difference in meaning between a coefficient and an exponent respectively:

$$3x = x + x + x.$$

$$x^3 = x \cdot x \cdot x.$$

If $x = 5$ in each case, $3x$ stands for the number 15, while x^3 stands for 125. If $x = 10$ in each case, $3x = 30$, while $x^3 = 1000$.

ORAL EXERCISES

1. What are the numerical coefficients in $4x$? $5a^2$? $3ax$? $4ac$? $3abc$?

2. What is meant by $3a$? $4x$? $5c$?

3. In $4a^2xy$, name the coefficient of a^2xy, xy, y, a^2x, and a^2y.

9. Use of parentheses and radical signs. If two or more numbers connected by signs of operation are inclosed in parentheses, the entire expression is treated as a symbol for a single number.

Thus $3(6+4)$ means $3 \cdot 10$, or 30; $(17-2) \div (8-3)$ means $15 \div 5$, or 3; $(5+7)^2$ means 12^2, or 144; and $6(x+y)$ means six times the sum of x and y.

As in arithmetic, the symbol for square root is $\sqrt{}$, and the symbol for cube root is $\sqrt[3]{}$.

The name **radical sign** is applied to all symbols like the following: $\sqrt{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, etc. The small figure in a radical sign, like the 3 in $\sqrt[3]{}$, is often called the **index**.

ORAL EXERCISES

Find the value of:

1. $2(3+4)$.
2. $4(7-2)$.
3. $(4+3)(5-2)$.
4. $(7-2)(8+3)$.
5. $\sqrt[3]{8}$.
6. $\sqrt{9+7}$.
7. $\sqrt{3^2+4^2}$.
8. $\sqrt[3]{4(7-5)}$.
9. $\sqrt[3]{(5+3)(6+2)}$.
10. $\sqrt{6^2+8^2}$.

NOTE. There has been a considerable variety in the symbols for the roots of numbers. The symbol $\sqrt{}$ was introduced in 1544 by the German Stifel, and is a corruption of the initial letter of the Latin word *radix*, which means "root." Before his time square root was denoted by the symbol ℞, used nowadays by physicians on prescriptions as an abbreviation for the word *recipe*. Thus $\sqrt[4]{5}$ would have been denoted by ℞^{4}5. Some early writers used a dot to indicate square root, and expressed $\sqrt{2}$ by $\cdot 2$. The Arabs denoted the root of a number by an arabic letter placed directly over the number.

ORAL EXERCISES

1. What are the numerical coefficients in $2x$? $3a^2$? $4xy$? $2ab$? $3\sqrt{a}$?

2. What are the exponents in $3a^2b$? $4a^2b^3$? $5a^3x^4$? $5x^2yz^3$?

3. What is the difference in meaning between the 4 in $4x$ and that in x^4?

4. What is meant by $2x$? $5a$? $8a$?

5. What is meant by $2a^2$? $3x^2$?

6. What is meant by $3(8+6)$? $2(9-4)$? $(7+3)(8-2)$? $(7+3)^2$? $\sqrt{3+6}$? $\sqrt{9+16}$? $\sqrt{100-64}$? $\sqrt[3]{35-8}$? $\sqrt[3]{100-36}$?

7. What is the numerical value of each expression in Exercise 6?

8. Read Exercises 1–16 on pages 15–16.

9. $3 \cdot 5^2 = ?$

10. $(8+2)^3 = ?$

11. $7(6-1) = ?$

12. $(4+3)(5+4) = ?$

13. $(5-1)(8+3) = ?$

14. $3(7-2)(5-3) = ?$

15. $\sqrt{5^2+12^2} = ?$

16. $\sqrt{(10+8)(10-8)} = ?$

In Exercises 17–33, replace *a* by 3 and *b* by 4 and find the value of the resulting expressions.

17. $3a^2-2b$.

18. $3b^2+8a$.

19. $\sqrt{b}$.

20. b^3.

21. $2a^2$.

22. $2a^2b$.

23. $4ab^2$.

24. $2a^2b^2$.

25. $3a^2b^3$.

26. $4ab+b^2$.

27. $\sqrt{3ab}$.

28. $\sqrt{a^2+b^2}$.

29. $\sqrt{3b^2+16}$.

30. $\sqrt{4a^2+7b}$.

31. $\sqrt[3]{2b}$.

32. $\sqrt[3]{ab+15}$.

33. $\sqrt[3]{a^2+b^2-17}$.

EXERCISES

Write, using algebraic symbols:

1. The sum of three times *a* and four times *b*.

2. Three times *a* subtracted from four times *b*.

3. The square of *a* subtracted from the square of *b*.

4. The cube of *b* subtracted from the square of *a*.

5. Two times *a* squared subtracted from three times *a* squared.

6. The quotient of *a* and *b*.

7. The product of four times *a* squared and *b*.

8. The sum of a and b divided by their product.

9. The product of a and $2b - c$.

10. The product of a and the sum of b and c.

11. The result of subtracting $a - b$ from $7x$.

12. The sum of the square root of $5a$ and the cube root of $7b$.

13. The product of $x - y$ and the square root of $7x$.

14. The square of the sum of a and b.

15. The square of b subtracted from a.

16. The quotient of three times a multiplied by the square of b, and four times c multiplied by the cube of a.

17. The sum of the quotients of a and $3x$, and $4y$ and c.

10. Order of fundamental arithmetical operations. If we read the expression $6 + 4 \cdot 9 - 12 \div 3$ from left to right, and perform each indicated operation as we come to its symbol, we obtain successively 10, 90, 78, and a final result of 26. If we perform the multiplication and division first, the expression becomes $6 + 36 - 4$, which equals 38. These results show that the value of the expression is determined largely by the order in which the operations are performed. It is customary to observe the

Rule. *In a series of operations involving addition, subtraction, multiplication, and division of arithmetical numbers, the multiplications and divisions shall be performed first in the order in which they occur. The additions and subtractions in the resulting expression shall then be performed in the order in which they occur or in any other order.*

If parentheses occur, each expression within parentheses should first be simplified in accordance with the preceding rule and the rule then applied to the whole.

EXAMPLES

Simplify:

1. $18 \div 2 + 5 - 4 \cdot 2.$

Solution. $18 \div 2 + 5 - 4 \cdot 2 = 9 + 5 - 8 = \mathbf{6}.$

2. $24 \div 8 \cdot 4 - 6 + 5 \cdot 2 - 7.$

Solution. $24 \div 8 \cdot 4 - 6 + 5 \cdot 2 - 7 = 3 \cdot 4 - 6 + 10 - 7 =$

$12 - 6 + 10 - 7 =$

$22 - 13 = \mathbf{9}.$

3. $4 \cdot 2 - 3 + 2(8 \cdot 4 - 12 \div 3 + 2 - 6) - 18 \div 2.$

Solution. $4 \cdot 2 - 3 + 2(8 \cdot 4 - 12 \div 3 + 2 - 6) - 18 \div 2 =$

$8 - 3 + 2(32 - 4 + 2 - 6) - 9 =$

$5 + 2(\quad 24 \quad) - 9 =$

$5 + 48 - 9 = \mathbf{44}.$

EXERCISES

Simplify the following:

1. $20 - 5 + 6 - 10.$

2. $16 - (8 - 2).$

3. $14 - (16 - 8) + (12 - 4).$

4. $6 \div 3 - 2.$

5. $8 \cdot 6 \div 3 - 10.$

6. $18 \div (2 \cdot 3).$

7. $(6 - 3) \cdot (17 - 2 \cdot 5).$

8. $23 - 2 \cdot 6 - 4 \div 2 + 16.$

9. $18 \div (9 - 3).$

10. $(10 - 3) \cdot (16 - 3 \cdot 2 + 8 \div 4).$

11. $14 - 3 \cdot (16 - 2 \cdot 5) \div 6 + 8 \cdot 2.$

12. $(18 - 2) - (4 + 2 \cdot 8 - 18 \div 9) \div 6.$

13. $(16 - 6) \cdot (18 - 8) \div 100 \cdot 5 - 5.$

14. $(5 + 3) \cdot (5 - 3) \div 4 - 3.$

15. $3^2 - 2 \cdot 3 \cdot 1 + 1^2.$

16. $8^2 - 2 \cdot 8 \cdot 3 + 3^2.$

17. $4 \cdot 3 - 2(6 - 2 \cdot 3) + 8.$

18. $8(12 + 4 + 3 - 1).$

19. $3 + 2 \cdot 4 + (3 + 2)4.$

20. $(8 - 2)3 + 8 - 2 \cdot 3.$

21. $3 \cdot 4 - 6 \cdot 0 + 2 \cdot 5 + 2 \cdot 7^2 - 2 \cdot 8.$

22. $(12 + 24 \times 18 \div 3 + 6) \cdot (24 \div 4 + 3 - 2).$

11. Evaluation of algebraic expressions. It is frequently necessary to find the numerical value of an expression for certain values of the letters involved. This process will be found useful in detecting errors made in the solution of equations. (See page 42.)

In practical affairs a working rule, a geometrical relation, or a scientific fact is often stated briefly and conveniently by means of an algebraic expression. Such expressions are frequently called formulas. The student will recall that arithmetic furnished many illustrations of their use. Thus, $A = \frac{ab}{2}$ is the formula for the area of a triangle; $I = P \cdot r \cdot t$ is the formula for simple interest; $A = \pi r^2$ is the formula for the area of a circle, etc.

In finding the numerical value of any literal expression the student should observe the following

Rule. *First, put in place of each letter its numerical value; second, simplify the result thus obtained.*

In any but the simplest expressions the student should *always* observe the two steps of the above rule separately in the order in which they are stated. To mix the two in an attempt to perform mentally both processes at once, is sure to result in many errors and consequent loss of time.

EXERCISES

In Exercises 1–16, let $a = 3$, $b = 1$, $c = 5$, $d = 7$, and $f = 2$. Substitute for each letter its numerical value, and then simplify the results according to the rule of page 13:

1. $4a^2 - 7b$.

Solution.

$$4a^2 - 7b = 4 \cdot 3^2 - 7 \cdot 1$$
$$= 36 - 7 = 29.$$

2. $4a + 3d$.

3. $ab + cd$.

4. $c^2 - 3ab$.

5. $abcd - 5f^2$.

6. $\frac{d + c}{f}$.

7. $\frac{cdf}{3f} + \frac{acf}{2a}$.

8. $\frac{12b}{f} - \frac{c}{b}$.

9. $\frac{1}{c} + \frac{1}{d} + \frac{1}{f}$.

10. $\frac{f^2}{ac} + \frac{6cd}{af}$.

11. $\dfrac{a^2+b^2+c^2+d^2}{a+b+c+d}$.

12. $5f^3+4f^2-4f-5$.

13. $3f^5-9f^4+11f^3-11f^2+13f-20$.

14. a^f.

15. d^f+f^a.

16. $\dfrac{d+f^c}{3c+a}$.

Find the numerical value of the following expressions when $a=4$, $b=0$, $c=5$, $d=7$, and $f=8$:

17. $\dfrac{4a+3b+2d}{c+f+2}$.

18. $\dfrac{b}{a+c+d}$.

19. $\dfrac{ab}{c}+\dfrac{bd}{a}+\dfrac{bf}{cd}$.

20. $\sqrt{a}+\sqrt{2f}$.

21. $c\sqrt{2af}$.

22. $\sqrt{c^2-a^2}$.

23. $(a+f)\cdot(c+d)$.

24. $ab(a+b)$.

25. $(a+c)\cdot(a+c)$.

26. $a+c(a+c)$.

27. $(a+c)\cdot a+c$.

28. $a+ca+c$.

29. $(f-a)^2$.

30. $(d-a)^3$.

31. $a^2-2ac+c^2$.

32. $a^2+2ab+b^2$.

33. $c^3-3c^2a+3ca^2-a^3$.

34. $d\sqrt[3]{2af}$.

35. $\sqrt{a^2+ac}$.

36. $\sqrt[3]{ad+36}$.

37. If $x=2$ and $y=3$, does $13x-5y=11$?

38. If $x=8$, does $7x-9=3x+25$?

39. Does $x^2-5x+6=0$, if $x=2$? if $x=3$? if $x=4$?

40. Does $x^2-7x+12=0$, if $x=2$? if $x=3$? if $x=4$?

41. Does $2x^2-5x-3=0$, if $x=4$? if $x=3$? if $x=\frac{1}{2}$?

42. The area of a triangle is given by the expression $A=\dfrac{ab}{2}$ in which A is the area, a is the altitude, and b is the base. Find the area of a triangle in which the altitude is 11 inches and the base 14 inches.

43. The area of a circle is given by the formula $A=\pi r^2$ in which A is the area, π is $\frac{22}{7}$ (approximately), and r is the radius. Find the area of a circle whose radius is (*a*) 7 inches; (*b*) $\frac{1}{2}$ inch.

44. The volume of a circular cylinder is given by the formula $V=\pi r^2h$ in which V is the volume, r is the radius of the base, and h is the altitude of the cylinder. Find the volume of a cylinder in which: (*a*) $r = 3$ inches and $h = 5$ inches; (*b*) $r = 4$ inches and $h = 7$ inches.

45. The distance a body falls from rest is given by the formula $s = \frac{at^2}{2}$ in which $s =$ the distance in feet, $a = 32$, and $t =$ the time in seconds. How far will a body fall in 5 seconds?

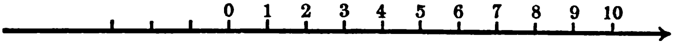
46. A balloonist drops a rock while crossing a river. He sees it strike the water 16 seconds later. How high was the balloon at the time?

47. The horse power of a certain kind of gasoline engine is given by the formula $H = \frac{d^2n}{6}$ in which H is the horse power, d is the diameter of each cylinder in inches, and n is the number of cylinders. Find the horse power of a four-cylinder engine in which the diameter of each cylinder is 4 inches.

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS

12. Addition and subtraction. Let us suppose that equal distances are taken on a line and the successive points of division are marked with the natural numbers as follows:



(*A*)

Such a scale of numbers may be used to illustrate both addition and subtraction as performed in arithmetic.

Thus, in adding 5 to 3 we may begin at 3 and count on 5 spaces to the right, obtaining the sum 8. We shall obtain the same result if we begin at 5 and count on 3 spaces to the right. This process may be stated in general terms thus:

Rule. *To add the number **a** to the number **b**, begin at **b** and count on **a** spaces to the right.*

In subtracting 4 from 7 we may begin at 7 and count off 4 spaces to the left, thus obtaining 3. This process may be stated in general terms thus:

Rule. *To subtract the number **a** from the number **b**, begin at **b** and count off **a** spaces to the left.*

If we attempt to subtract 5 from 4 by the preceding rule, we arrive at the first point of division to the left of zero. Arithmetic has no number to represent such a result; in fact, the subtraction of 5 from 4 is there regarded as impossible. Arithmetically speaking, such a subtraction cannot be performed. We can, however, subtract 4 of the 5 units from the 4 units, leaving 1 unit unsubtracted.

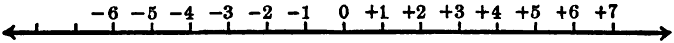
Now in algebra it is both convenient and necessary to speak of subtracting a greater number from a less, and to call the portion of the greater number which is unsubtracted, the remainder. The fact that such a subtraction is incomplete is indicated by writing a minus sign before the result; thus, $4-5=-1$. Hence the first point of division to the left of zero may be thought of as corresponding to -1. Similarly, $3-5=-2$; and to -2 may correspond the second point of division to the left of zero.

In like manner $5-8=-3$, which corresponds to the third point to the left of zero. In the same way the fourth point of division to the left of zero would correspond to -4, the fifth point to -5, etc.

Such numbers as -1, -2, -3, etc. are called **negative** numbers. The minus sign is never omitted in writing a negative number, though a letter, as x, may denote one.

In opposition to negative numbers the ordinary numbers of arithmetic are called **positive** numbers. If a number has no sign before it, or a plus sign, it is a positive number.

The relative order of positive and negative numbers is indicated in the following scale:



(*B*)

ORAL EXERCISES

Perform the following additions and subtractions by counting along the preceding scale according to the rules on page 18:

1. Add 4 to 3.

2. Add 4 to -2.

3. Add 6 to -3.

4. Add 3 to -3.

5. Add 2 to -5.

6. Add 5 to -7.

7. Subtract 2 from 5.

8. Subtract 5 from 2.

9. Subtract 6 from 3.

10. Subtract 4 from -3.

11. Subtract 2 from -4.

12. Subtract 2 from -3.

13. Practical use of positive and negative numbers. The scale (B) of positive and negative numbers could be used to measure many of the things with which we come in daily contact. In fact, a practical equivalent is already in use in many instances. Thus, in graduating a thermometer a certain position of the mercury is taken as zero, and the degrees are marked both above and below this point. Hence a temperature reading of 18° is indefinite unless accompanied by the words *above zero* or *below zero*. Usually $+18°$ is taken to indicate the former, while $-18°$ indicates the latter.

Similarly any point on the earth's equator is in zero latitude. Latitude 40° N. means 40° north of the equator. In like manner 30° S. means 30° south of the equator. Obviously $+40°$ and $-30°$ might be used to convey the same ideas.

In general all concrete uses of positive and negative numbers occur in connection with magnitudes which may be regarded as opposite in sense; as, for example, money in bank and an overdrawn account, distances measured in opposite directions from a fixed point, and time measured in the future and in the past from a certain instant.

EXERCISES

1. If the temperature is now $+14°$, what will represent the temperature after a fall (*a*) of 5°? (*b*) of 10°? (*c*) of 18°?

2. If the temperature is now $-16°$, what will it be after a rise (*a*) of 7°? (*b*) of 12°? (*c*) of 25°?

3. In the preceding exercise change the word *rise* to *fall* and then answer.

4. A ship sails south from latitude $+13°$ to latitude $-7°$. If one degree is 69 miles, how far did it sail?

5. A ship sails south from latitude + 20° at the rate of 5° daily. In what latitude is it at the end of each of 6 days? After how many days will it reach latitude − 15°?

6. A man's property is worth $4200 and his debts amount to $2300. How can positive and negative numbers be used to represent (*a*) each of these amounts? (*b*) the man's financial standing?

NOTE. So far as is known the first explanation of positive and negative numbers was by means of the illustration of assets and debts. This is found in the writings of the Hindus before 700 A.D., long before negative numbers were accepted as having any definite meaning. In the use of this illustration the Hindus were nearly a thousand years in advance of the times.

7. If debts and property be reversed in Exercise 6, what would be the answer to (*a*) and (*b*)?

8. The temperature at 6.00 A.M. was − 12°. During the morning it rose at the rate of 3° an hour. What was the temperature at 9.00 A.M.? 10.00 A.M.? 12.00 M.?

14. Addition of positive and negative numbers. As we have seen, subtraction by the use of scale (*B*) is performed by counting spaces to the left. Now a negative number represents an unperformed subtraction; therefore to add a negative number to another number means to perform this subtraction.

For example, in subtracting 8 from 5, − 3 was obtained by beginning at 5 and counting 8 spaces to the left, arriving at 3 to the left. Hence when we wish to add − 3 to any number, we count off 3 spaces to the left from that number.

To add − 8 to 24 we begin at 24 and count off 8 spaces to the left, obtaining 16 as the result; that is,

$$+24+(-8)=+16.$$

Similarly, to add -6 to -4 we begin at -4 and count 6 spaces to the left, obtaining -10 as the result; that is,

$$-4+(-6)=-10.$$

Hence, in general, to *add* a *negative* number n to a given number, begin at the given number and count off n spaces to the *left*.

The **numerical** or **absolute** value of a number is its value without regard to sign.

Thus the absolute values of -3, -5, and $+7$ are 3, 5, and 7 respectively. It should be noted that two different numbers, as $+6$ and -6, may have the same absolute value.

ORAL EXERCISES

By the use of scale (*B*), page 19, add the following:

1. $+5, +3$.
2. $+5, -3$.
3. $-5, +3$.
4. $-5, -3$.
5. $7, -3$.
6. $-7, -3$.
7. $-6, -2$.
8. $6, -4$.
9. $4, -7$.
10. $-3, -5$.
11. $-4, +4$.
12. $-6, +8$.

The preceding exercises illustrate the correctness of the following working rules:

I. To add two or more positive numbers, find the sum of their absolute values and prefix to this sum the plus sign.

II. To add two or more negative numbers, find the sum of their absolute values and prefix to this sum the minus sign.

III. To add a positive and a negative number, find the difference of their absolute values and prefix to the result the sign of the one which has the greater absolute value.

These rules are of vital importance, as they are of almost constant use throughout the whole of algebra. Rule III is the one in the use of which errors are most likely to occur.

Hence the student will save time if he masters Rule III at *this* point.

The **algebraic sum** of two or more numbers is the number obtained by adding them according to the preceding rules.

The algebraic sum of two numbers is often different from the sum of their absolute values; for example, the algebraic sum of $+9$ and -5 is $+4$, while the sum of their absolute values is 14.

Hereafter the word *add* will mean *find the algebraic sum.*

ORAL EXERCISES

Perform the indicated additions:

1. $+5+(+7)$.

2. $-5+(-7)$.

3. $+5+(-7)$.

4. $-5+(+7)$.

5. $+8+(-5)$.

6. $-8+(+5)$.

7. $-12+(-9)$.

8. $-6+(+6)$.

9. $-6+(-6)$.

10. $-4+(+3)+(+6)$.

11. $3+(-7)+(+5)+(-4)$.

12. $8+(-2)+(-4)+(+6)$.

Answer the questions asked in the following:

13. $7+?=9$.

14. $7+?=2$.

15. $8+?=12$.

16. $8+?=5$.

17. $-8+?=-10$.

18. $-8+?=-6$.

19. $-10+?=-16$.

20. $-10+?=3$.

21. $12+?=5$.

22. $-12+?=4$.

15. Subtraction of positive and negative numbers. If we wish to subtract 7 from 12, we may do so by answering the question, "What number added to 7 gives 12?" By answering a similar question we can subtract 8 from 15, or 25 from 43, or any number a from another number b.

Exercises 13–22, p. 23, are therefore exercises in subtraction, for each asks a question similar to the one in the first sentence of this paragraph.

This point of view brings out the relation that the operation of subtraction bears to that of addition.

ORAL EXERCISES

Perform the following subtractions by answering in each case the question, "What number added to the first number gives the second number?"

Subtract:

1. 5 from 9.

2. 9 from 14.

3. 8 from 5.

4. 13 from 7.

5. -5 from -8.

6. -5 from 9.

7. 5 from -10.

8. -6 from -4.

9. 12 from -18.

10. -25 from 14.

11. In Exercises 1–10, change the sign of the subtrahend (if $+$, to $-$; if $-$, to $+$) and then add the subtrahend to the minuend. Are the answers the same as were obtained before?

The results obtained in Exercise 11 illustrate the following important principles:

I. Subtracting a positive number is the same in effect as adding a negative number of the same absolute value.

To illustrate: a decrease of $100 in a man's assets is equivalent to an increase of $100 in his liabilities, provided we consider his real financial standing in each case.

II. Subtracting a negative number is the same in effect as adding a positive number of the same absolute value.

To illustrate: a decrease of $75 in a man's liabilities is equivalent to an increase of $75 in his assets, as far as his net financial standing is concerned.

Hence, for the subtraction of positive and negative numbers, we have the

Rule. *Change the sign of the subtrahend (if +, to −; if −, to +). Then find the algebraic sum of the subtrahend (with its sign changed) and the minuend.*

This rule really turns algebraic subtraction into algebraic addition and marks one of the most important distinctions between the operations of arithmetic and those of algebra. For this reason little genuine progress in algebra can be made till the student becomes proficient in the use of the method stated in the rule.

ORAL EXERCISES

Subtract the second number from the first in Exercises 1–16:

1. $8, +5$.
2. $+8, -5$.
3. $+5, -8$.
4. $+5, +8$.
5. $14, +9$.
6. $14, -9$.
7. $-14, +9$.
8. $-14, -9$.
9. $+6, +6$.
10. $-6, -6$.
11. $6, -6$.
12. $-6, +6$.
13. $-15, -19$.
14. $0, +1$.
15. $-1, -1$.
16. $1, -2$.
17. $12-(+3)-(+2)=?$
18. $-10-(+3)-(+2)=?$
19. $-12-(-3)-(-2)=?$
20. $17-(-5)-(+7)=?$

Supply the missing numbers in:

21. $+7+?=10$.
22. $-6+?=-10$.
23. $-5+?=0$.
24. $+6+?=0$.
25. $+4+?=4$.
26. $-8+?=-3$.
27. $-9+?=-5$.
28. $-7+?=7$.
29. $9-?=5$.
30. $-7-?=-5$.
31. $+5-?=-9$.
32. $-7-?=4$.
33. $-5-?=0$.
34. $2-?=0$.
35. $4-?=16$.
36. $6+?=-3$.

Simplify:

37. $12+(7)-(5)$.

38. $12-(-3)+6$.

39. $12-(-4)+(-5)$.

40. $18+(-6)-(+7)$.

41. $-16+(-10)-(+12)$.

42. $-13-(8)+(-14)$.

16. Multiplication of positive and negative numbers. In arithmetic, multiplication was defined as the process of taking one number, the multiplicand, as many times as there are units in another number, the multiplier. The original signification of *times* made this definition meaningless when the multiplier was a fraction; for in $8\times\frac{3}{5}$, 8 could not be added $\frac{3}{5}$ of a time. The definition was then extended and the product of 8 multiplied by $\frac{3}{5}$ was defined to mean 8 multiplied by 3 and the result divided by 5; that is, $8\times\frac{3}{5}$ means $\frac{8\times 3}{5}$.

Since algebra deals with both positive and negative numbers, we must now extend the arithmetical definition of multiplication and define what sign the product shall have in each of the four cases which may possibly arise:

$$(+4)\cdot(+3)=?\qquad (+4)\cdot(-3)=?$$
$$(-4)\cdot(+3)=?\qquad (-4)\cdot(-3)=?$$

By $(+4)\cdot(+3)$ we mean that $+4$ is to be added three times:

$$(+4)+(+4)+(+4)=+12;$$

that is,

$$(+4)\cdot(+3)=+12.$$

Similarly $(-4)\cdot(+3)$ means

$$(-4)+(-4)+(-4)=-12;$$

that is,

$$(-4)\cdot(+3)=-12.$$

In $(+4)\cdot(-3)$ we mean that 4 is to be subtracted three times. This is the same as subtracting 12 once.

Therefore

$$(+4)\cdot(-3)=-12.$$

Lastly, by $(-4)\cdot(-3)$ we mean that -4 is to be subtracted three times. This is the same as subtracting -12 once, and subtracting -12 once is the same as adding $+12$. Therefore

$$(-4)\cdot(-3)=+12.$$

Summing up,

$$(+4)\cdot(+3)=+12. \qquad (+4)\cdot(-3)=-12.$$
$$(-4)\cdot(+3)=-12. \qquad (-4)\cdot(-3)=+12.$$

Or, in general terms,

$$+a\times+b=+ab.$$
$$-a\times+b=-ab.$$
$$+a\times-b=-ab.$$
$$-a\times-b=+ab.$$

Therefore we have the

Rule. *The product of two numbers having like signs is a positive number, and the product of two numbers having unlike signs is a negative number.*

ORAL EXERCISES

Find the products of the following:

1. $+3, +5$.
2. $+4, +12$.
3. $-5, +6$.
4. $+6, -6$.
5. $-7, +8$.
6. $-7, -4$.
7. $-12, +9$.
8. $+6, -4$.
9. $-7, -6$.
10. $+5, -10$.
11. $+0, +4$.
12. $-7, 0$.
13. $+4, -7, +6$.
14. $+4, -5, -6$.
15. $-4, -5, -3$.
16. $12, +0, -5$.
17. $9, -10, -0$.
18. $-4, +7, -6$.
19. $-3, -2, -5$.
20. $2, -3, +9$.

NOTE. The famous German mathematician Leopold Kronecker (1823–1891) once observed that "the good Lord made the positive integers, but man is responsible for all the rest of the numbers."

This expresses the truth about numbers as accurately as one can in a single sentence. We count objects from our earliest years, and so use the positive integers naturally. It is only when we come to study mathematics that the necessity for any other kind of numbers is forced upon us. Here we see that negative numbers are a great convenience if we wish to represent the relations between objects where oppositeness in any of its many forms is involved. But the artificial character of negative numbers delayed their intelligent use for many hundred years. To be sure, the Hindus said that "the square of negative is positive," but the statement probably did not mean anything to those who read it. It was not until after the time of Descartes (see p. 210) that the rules for operating on negative numbers were understood, even by great mathematicians.

17. Division of positive and negative numbers. When 18 is divided by 9 the result is 2. Here 18 is the dividend, 9 the divisor, and 2 the quotient. The three are connected by the following relation, which holds both for arithmetic and algebra:

$$\text{quotient} \times \text{divisor} = \text{dividend}.$$

We can see that 2 is the correct value of $18 \div 9$, because $2 \times 9 = 18$. This simple test will be applied to determine whether the quotient is a positive or a negative number. All the cases which may arise are represented by the four following questions:

(*a*) $+18 \div +9 = ?$ (*c*) $+18 \div -9 = ?$
(*b*) $-18 \div +9 = ?$ (*d*) $-18 \div -9 = ?$

These questions are answered as follows:

(*a*) $+18 \div +9 = +2$ because $+2 \cdot +9 = +18$.
(*b*) $-18 \div +9 = -2$ because $-2 \cdot +9 = -18$.
(*c*) $+18 \div -9 = -2$ because $-2 \cdot -9 = +18$.
(*d*) $-18 \div -9 = +2$ because $+2 \cdot -9 = -18$.

In (*a*) and (*d*) the dividend and divisor have *like* signs and the sign of the quotient is *plus*. In (*b*) and (*c*) the dividend and divisor have *unlike* signs and the sign of the quotient is *minus*.

Therefore we have the

Rule. *The quotient of two numbers having like signs is a positive number, and the quotient of two numbers having unlike signs is a negative number.*

The result of multiplication by zero is given a definite meaning in arithmetic and algebra, namely zero; but in both subjects *division by zero is always excluded.* If zero were used as a divisor, numerous contradictions would arise of which the following is an illustration:

Obviously, $0 \cdot 4 = 0,$
and $0 \cdot 6 = 0.$
Therefore $0 \cdot 4 = 0 \cdot 6.$
Dividing each by zero, $4 = 6,$
which is false.

NOTE. The Hindus were the first to express the laws that govern the operations with the number 0. In fact, they were the first to have such a symbol. In the twelfth century a Hindu writer states that $a + 0 = a$, that $\sqrt{0} = 0$, and that $0^2 = 0$. Of course he did not express himself in terms of these symbols, but in the notation of his time and country.

ORAL EXERCISES

Divide the first number by the second in Exercises 1–9:

1. $+10, +2.$
2. $-10, -5.$
3. $-15, +3.$
4. $+14, -2.$
5. $-18, -3.$
6. $-7, +7.$
7. $0, +5.$
8. $0, -5.$
9. $-3, -3.$

10. $+36 \div (+3) \div (-2) = ?$
11. $+45 \div (-5) \div (-3) = ?$
12. $-64 \div (+4) \div (-2) = ?$
13. $+96 \div (-6) \div (+8) = ?$
14. $72 \div (+6) \div (-4) = ?$
15. $60 \div (-5) \div (-12) = ?$
16. $48 \div 2 \div (-4) = ?$
17. $\dfrac{-20}{?} = 4.$
18. $\dfrac{-12}{?} = -2.$
19. $\dfrac{+36}{?} = 4.$

18. Omission of the plus sign before a number. If the first of several numbers connected by either plus or minus signs is a positive number, its sign is omitted; thus $+4-3+6$ is written $4-3+6$. If the sign of the 4 had been negative, the minus sign could not properly have been omitted.

19. Omission of the sign of multiplication. If each of two or more numbers be inclosed in a parenthesis and the parentheses be written one directly after the other with no sign of operation between them, the sign of multiplication is always understood; thus $(6)(3)$ means $6 \cdot 3$. Similarly $(6)(5)(2) = 6 \cdot 5 \cdot 2$.

MISCELLANEOUS ORAL EXERCISES

Simplify the following:

1. $(7)+(5)$.
2. $(7)-(5)$.
3. $(7)+(-5)$.
4. $(7)-(-5)$.
5. $(-7)+(5)$.
6. $-7+5$.
7. $(-9)-(4)$.
8. $-9-4$.
9. $-11+(-13)$.
10. $-6-(-10)$.
11. $-6+10$.
12. $8+(-10)$.
13. $12-18$.
14. $-18-12$.
15. $15-14$.
16. $+7-0$.
17. $0-3$.
18. $(-3)(6)$.
19. $(-5)6$.
20. $(7)(-5)$.
21. $8(-3)$.
22. $(-5)(-12)$.
23. $-3(-8)$.
24. $-5 \cdot 4$.
25. $5 \cdot 0$.
26. $0 \cdot (-9)$.
27. $4 \cdot 8$.
28. $-3 \cdot 6$.
29. $12 \div (-2)$.
30. $-12 \div 2$.
31. $-39 \div (-3)$
32. $45 \div (-15)$.
33. $0 \div (-6)$.
34. $0 \div 3$.
35. $-27 \div 9$.
36. $3-5+6$.
37. $-4+6-2+1$.
38. $-4+6+2-1$.
39. $2-3+4-5-6$.

Add:

40.	**41.**	**42.**	**43.**
7	6	−8	4
−2	−2	6	−9
3	−3	2	−3
−5	4	−5	6

Simplify:

44. $3 \cdot 6 \div 3$.

45. $-4(7) \div (-2)$.

46. $3(-6) \div 2$.

47. $4 \cdot 6(-8) \div (-16)$.

48. $18 \div (-3) \cdot 6 \div 4$.

49. 3^2.

50. $(-3)^2$.

51. 2^3.

52. $(-2)^3$.

53. $(-4)^3 + (4)^2$.

54. $-(-5)^2$.

55. $-(-5)^3$.

56. $-2(3)^2(-2)$.

57. $-3(-2)(-3)^2$.

58. $-2(4)^2(-5)$.

59. $2(-4)^2 \cdot 5$.

MISCELLANEOUS EXERCISES

Simplify the following:

1. $4^2 - (-2)^2$.

2. $3^3 - (-2)^3$.

3. $(5-3)(3+2)$.

4. $(6-2)(7-3) \div (4-9)$.

5. $(6-\frac{1}{2})(5+\frac{1}{3})$.

6. $(-1)^2 + (-1)^3 + (-2)^2 + (-2)^3$.

7. $9 + 3 \cdot 2 + 18 \div (-3)$.

8. $5^2 - 4 \div (-2) \div 6(-3)$.

9. $3 \cdot 6 \div 9 - 2 \cdot 6 \div 4 + (-3)^2$.

10. $3^2 + 2 \cdot 3 \cdot (-4) \div (-4)^2$.

11. $(+2)^2 - 2(2)(-3) + (-3)^2$.

12. $3(4)^2(-5) - (-5)^3$.

13. $3^3 + 3(3)^2(-2) + 3(3)(-2)^2 + (-2)^3$.

14. $4^3 - 3(4)^2(-3) + 3(4)(-3)^2 - (-3)^3$.

15. $3^3 - 3(3)^2(0) + 3(3)(0)^2 - 0^3$.

16. $5^2 + 3(5)^2(-4) + 3(5)(-4)^2 - (-4)^3$.

If $x = 3$ and $y = -2$, find the value of:

17. y^2. **19.** y^4. **21.** $2y^2$. **23.** $5x^2y^2$.
18. y^3. **20.** y^5. **22.** $2y^3$. **24.** $4x^2y^4$.

25. $x^2 - y^2$. **28.** $x^2 - 2xy + y^2$.
26. $x^3 - y^3$. **29.** $(x + y)(x - y)$.
27. $x^2 + 2xy + y^2$. **30.** $x^3 + 3x^2y + 3xy^2 + y^3$.

31. Does $4x - 2 = 2x + 8$, if $x = 5$?

32. Does $3x - 5 = 2x + 8$, if $x = -9$?

33. Does $x^2 - x - 12 = 0$, if $x = 4$? if $x = -8$? if $x = -4$?

34. Does $3x^2 + 19x = 14$, if $x = \frac{2}{3}$? if $x = 2$? if $x = -7$?

35. At 7.00 A.M. on a certain day the thermometer registered 27 degrees above zero. The mercury then fell at the rate of 3 degrees per hour. What was the temperature at noon? at 4.00 P.M.? at 5.00 P.M.? When was the temperature -12?

36. A Zeppelin rises 1800 feet from its position, then falls 1200 feet, rises again 1750 feet, falls 400 feet, and then rises 820 feet. How many feet higher or lower is it than at first?

37. Euclid lived about 300 B.C. Sir Isaac Newton died in 1727 A.D. If dates before Christ are considered negative and those after Christ are considered positive, how might these dates be written?

38. What is the meaning of the date -450? of $+1917$? What is the difference in time between the two?

39. A boat is traveling 12 miles per hour. A man on its deck is walking 3 miles per hour. Using positive and negative numbers, represent the rate at which he approaches his destination when he walks toward the bow and when he walks toward the stern.

40. A balloon capable of supporting 500 pounds is held down by 10 men whose average weight is 150 pounds. Using positive and negative numbers, represent the weight of the balloon, of the men, and of the balloon and the men together.

CHAPTER III

ADDITION

20. Monomials. A number symbol which is not the indicated algebraic sum of two or more number symbols is called a **term** or **monomial.**

Thus 5, $-a$, b^4, a^2x, and $-4cy^2$ are terms. Frequently, where no confusion would arise, expressions like $(a+b)$, $3(x-y)$, $5\sqrt{x^3}$, and $\sqrt{a-x}$ are called terms, for in such cases one's thought is centered not on the parts of which the expression is composed but on a single number for which the whole stands.

21. Similar terms. Terms that are alike in every respect except their coefficients are called **similar.**

Thus 3, -7, and 9 are similar terms, as well as $\sqrt{2}$ and $3\sqrt{2}$. Also a, $4a$, and $-10a$ are similar terms, as are a^2x, $-3a^2x$, and $7a^2x$.

22. Addition of similar terms. We have already learned that $6a+3a=9a$, $6a+(-3a)=3a$, and $5xy+6xy=11xy$. In like manner the sum of $8y$, $-3y$, $2y$, and $-y$ is $6y$. The terms $-y$, x, ay, and $-c^2x$ are equivalent to $-1y$, $+1x$, $+1ay$, and $-1c^2x$ respectively.

Thus, for adding similar terms we have the

Rule. *Find the algebraic sum of the numerical coefficients and prefix this result to the common literal part.*

ORAL EXERCISES

Find the sum of:

1. $+8, -4$.

2. $+9, -12$.

3. $-6, -9$.

4. $-6a, -4a$.

5. $+12a, -7a$.

6. $-9a, +3a$.

7. $-2a, +3a$.

8. $+8x, -3x$.

9. $+5x, -7x$.

10. $-11x, +14x$.

11. $-12ac, -4ac, +9ac$.

12. $+ax, -3ax, +5ax$.

13. $+4ax$, $-ax$, $-7ax$.

14. $+6ax$, $-ax$, $-3ax$, $+ax$.

15. $+4ac$, $-3ac$, $+12ac$, $-8ac$.

16. $+3ac$, $-12ac$, $-10ac$, $+15ac$.

Combine:

17. $ax-2ax+8ax-3ax$.

18. $4am-6am-10am+8am$.

19. $5am-3am+12am-7am$.

20. $6am-7am+3am-2am$.

21. $-3ax-4ax+7ax-12ax$.

22. $-ac+4ac+ac-2ac+12ac$.

23. $4ax-10ax-12ax+7ax$.

24. $3ax-5ax+7ax-9ax+8ax$.

23. Order in adding terms. Obviously $2+3=3+2$, and $2-3+5=2+5-3=-3+5+2$, etc. This illustrates the law that in addition the terms may be arranged and added in any order. Hence $6d+7c=7c+6d$, and the sum of 3 and x is either $x+3$ or $3+x$; also $a+b=b+a$, and $a+b+c=b+c+a=c+a+b$.

24. Addition of dissimilar terms. Algebraic expressions for the sum of two terms which are not similar, such as $6d$ and $7c$, are obtained by writing them one after the other with a plus sign between them; thus, $6d+7c$. The addition of $6d$ and $-7c$ is indicated by writing $6d+(-7c)$, or, more briefly, $6d-7c$. Similarly the sum of $3x$, $-2y$, and $-7z$ may be written $3x+(-2y)+(-7z)$, or, omitting the parentheses and the unnecessary signs, $3x-2y-7z$.

Thus, for adding dissimilar terms we have the

Rule. *Write the terms one after another in any order, giving to each its proper sign.*

If similar and dissimilar terms are to be added, the two preceding rules must both be applied.

EXERCISES

Find the sum of:

1. $a,\ 4\,b,\ -c.$
2. $4\,x,\ -2\,b,\ 5\,y,\ 10.$
3. $3\,ab^2,\ 2\,bx,\ -cy,\ 4\,a^2b.$
4. $5\,x^3y,\ -5\,xy^3,\ c^3y,\ -2\,cy^3.$
5. $6\,x,\ -3\,a,\ 2\,b,\ -5\,x,\ 3\,y.$
6. $5\,a,\ -4\,b,\ +3\,c,\ 7\,b,\ -2\,c^2.$
7. $3\,a^2,\ +2\,b^2,\ -7\,c^2,\ -4\,b^2,\ 5\,a^2.$
8. $5\,a^3b,\ -5\,ab^3,\ -3\,a^2b,\ 3\,ab^2,\ 4\,a^3b,\ 2\,ab^3.$

Simplify:

9. $11+5-9-3+16-25.$
10. $16\,ac-9\,xy+5\,ac-2\,xy-6\,ac+11\,xy.$
11. $7\,x+6\,a-15\,a-8\,x+3\,x.$
12. $12\,y-17\,y+10\,b+20\,y-25\,b.$
13. $4\,ab-8\,xy-12\,ab+15\,ab-xy.$
14. $14\,x^2-13\,y^2+x^2-5\,x^2+y^2.$
15. $5\,z^2-4\,y^2-11\,y^2+z^2-7\,y^2.$
16. $7\,c^2d-5\,a^2b+6\,c^2d-a^2b+9\,a^2b.$
17. $-4\,a^2b^2+6\,x^2y^2-15\,a^2b^2+3\,x^2y^2+0\,a^2b^2.$
18. $-b^3-19\,a+17\,b^3+b^3-0\,b^3+13\,a.$
19. $12\,a^3b+6\,c^2d-a^3b+16\,a^3b-13\,a^3b-25\,c^2d.$
20. $11\sqrt{a}-16\sqrt{a}+21\sqrt{a}-\sqrt{a}.$
21. $3\sqrt{x-y}-\sqrt{x-y}+8\sqrt{x-y}-7\sqrt{x-y}.$
22. $3(a+b)-5(a+b)+8(a+b).$
23. $-7(a-2\,b)+(a-2\,b)+12(a-2\,b).$

25. Addition of polynomials. A **polynomial** is an algebraic expression consisting of two or more terms.

It is not usual to call an expression a polynomial if any of its terms contain a letter under a radical sign. Thus we shall not call expressions like $\sqrt{x-3}+4$ polynomials.

A **binomial** is a polynomial of two terms.

A **trinomial** is a polynomial of three terms.

EXAMPLE

Add the following polynomials: $4a - 6b - a^2c$; $3b + 4a^2c$; $-3a - 7a^2c + 10$; $5a + 3b - 6$.

Solution.

$$\begin{array}{rrrrr}
 & +4a & -6b & -\ a^2c & \\
 & & +3b & +4a^2c & \\
 & -3a & & -7a^2c & +10 \\
 & +5a & +3b & & -\ 6 \\
\hline
\textit{Sum,} & 6a & & -4a^2c & +\ 4
\end{array}$$

For the addition of polynomials we have the

Rule. *Write similar terms in the same column.*

Find the algebraic sum of the terms in each column and write the results in succession with their proper signs.

A **check** on an operation is another operation which tests the correctness of the first.

For example, in arithmetic the result of division is checked by multiplication; thus the check for $132 \div 6 = 22$, is $22 \cdot 6 = 132$.

Addition in arithmetic is usually checked by adding the columns in the opposite direction. In the two cases the mind deals at every step with different numbers, and if the final result is the same, its correctness is more probable than after either addition alone. But for arithmetical addition, and for many of the operations of algebra, no really satisfactory check exists.

In elementary algebra long columns in addition do not occur. Where columns contain three or more numbers they may be added term by term upward or downward. If the numbers in a column are not all of the same sign, we can partially check results by adding the positive and the negative numbers in such columns separately and finding the algebraic sum of the two results.

EXERCISES

Add the following polynomials and check the results:

1. $x + 3$, $2x - 6$, and $3x - 5$.

2. $x - y - 3$, $2x + y - 8$, and $7x - 4y + 10$.

3. $x + y + z$, $x - 3y + 4z$, and $3x + 4y - 7z$.

4. $2x + 5y - z$, $3x - 8y + 6z$, and $x - y - z$.

5. $3x - 1$, $4 - 2x$, $6x - 8$, and $3x + 3$.

6. $x^2 - 9$, $16 - 4x^2$, $8x^2 + 4$, and $3 - 2x^2$.

7. $5x + 5y$, $4x - 9y + 6z$, and $3x - 3y - 3z$.

8. $7x - 6y + 3z$, $5x - 4z$, and $2x + 6y - 5z$.

9. $x^2 - x - 2$, $3x^2 - 4x + 5$, and $5x^2 + 8x - 7$.

10. $x^2 - 4x + 1$, $3x - 2x^2 + 8$, and $10x - 11x^2 - 18$.

11. $3c - 4c^2 - 5$, $c^2 + c + 1$, $4c - 2c^2 + 3$, and $5 - 2c - 5c^2$

12. $4x - 5y - 5z$, $6y - 2z$, and $7x - 6y - 9z$.

13. $x + 2z + 3y$, $y - 3z + x$, and $z - 2x - 4y$.

14. $5x - 6y + 7z$, $2y - 11z + x$, and $9z - 5y$.

15. $8a - 7b - 6c$, $5c - 4a - 3b$, and $3b + 7c$.

16. $9ac - bc$, $7ab - 3ac$, and $-12ab - ac$.

17. $a^2 - 4a + 10$, $5a - 6a^2 + 4$, and $3a - 16 + 2a^2$.

18. $9 - a^2 + 3a$, $-13 + 6a^2 - 4a$, and $3a^2 - a$.

19. $a - x - 3c$, $4 - 7x + 3a$, $7c - 2a + 10$, and $a - c - 2x - 11$.

20. $a + x + c + 5$, $2a + 2c + 10 + 2x$, $a - c - 7 - x$, and $7 - 3x - 2a + 4c$.

21. $x - 1 - 3c$, $x - a - 2c$, $a - 3c - 5$, and $2x - 7 + 11a$.

22. $2x + 3y + 5z$, $x - y + 3z$, and $3x - 2y + 10z$.

23. $4x - 20z$, $8x + 9y$, and $5z - 3y$.

24. $a - 3(x - y) + z$, $5 - 10a + 4(x - y)$, and $-2(x - y) + 6$.

25. $a + d + 2(b - c)$, $7(b - c) + 6d - 12a$, and $11a - 5(b - c)$.

Combine similar terms in the following polynomials:

26. $7a^2 - 13b^2 + 12c^2 + 15b^2 - a^2 - 7c^2 + 3b^2 + 5c^2.$

27. $x^2 + 2xy + y^2 - 5xy - 4x^2 - y^2 + 16x^2 - 8xy + y^2.$

28. $5ab - a^2 + b^2 + 4ab - 9b^2 + 5a^2 - 2ab - 2b^2.$

29. $5x^2 - 6x + 11 - 4x - 8 - 3x^2 + 7x - 18 + 13x.$

30. $12c^2 - 10bc + 8b^2 + bc - 6b^2 - c^2 + c^3 - 11c^3.$

31. $4x^2y - xy + y^2 - 3xy + 4xy^2 - 2x^2y + 4y^2 + 3xy^2 + x^2y.$

32. $\frac{1}{2}a + \frac{1}{3}b - \frac{1}{4}c + c + \frac{1}{2}b - \frac{2}{3}a - b + \frac{1}{2}c + \frac{3}{4}a + 7.$

The sum of $5x$ and $2x$ may be written $(5 + 2)x$. This is not usual or necessary, as 5 and 2 can be combined and the result written $7x$. In adding $5x$ and ax, however, the 5 and the a cannot be combined and the result expressed by a single character, so the sum is written $(5 + a)x$. Similarly, $ax - 3x = (a - 3)x$, $ax + x = (a + 1)x$, and $ax - bx + x = (a - b + 1)x$. Also $a(x + y) + b(x + y) = (a + b)(x + y)$.

Write so that x will have a polynomial coefficient:

33. $ax + 3x.$ **35.** $3cx - x.$ **37.** $2ax - 3x + bx.$

34. $2ax + x.$ **36.** $ax + bx + cx.$ **38.** $3ax - 4cx + x.$

39. $3ax - bx - x + a^2x.$ **40.** $bx - 5cx - x - 4bx.$

Write so that the binomial will have a polynomial coefficient:

41. $a(b + c) + 3(b + c).$ **42.** $4(a - x) - 5b(a - x).$

43. $8a(a + 3b) - 1(a + 3b).$

44. $7b(x^2 + y^2) - a(x^2 + y^2) + (x^2 + y^2).$

CHAPTER IV

SIMPLE EQUATIONS

26. Definitions. An **equation** is a statement of equality between two equal numbers or number symbols.

Thus $2 = 5 - 3$, $a - 2b = 3a + b - 2a - 3b$, $4x = x + 12$, and $x^2 - 5x + 6 = 0$ are equations.

The part of an equation on the left of the equality sign is called the *first* or *left* member, that on the right, the *second* or *right* member.

In an equation a letter whose value is sought is called the *unknown letter*, or simply the *unknown.*

The process of finding the value of the unknown letter in an equation is called *solving the equation.*

27. Axioms. An **axiom** is a statement whose truth is accepted without proof.

In the solution of equations constant use is made of four axioms.

Axiom I. *If the same number is added to each member of an equation, the result is an equation.*

Thus, adding 5 to each member of $x - 5 = 7$ gives $x = 12$. Axiom I states that if this addition is performed, the result is true. Hence the original equation is solved.

Axiom II. *If the same number is subtracted from each member of an equation, the result is an equation.*

Thus, subtracting 4 from each member of $x + 4 = 10$ gives $x = 6$. Axiom II states that if this subtraction is performed, the result is true.

Axiom III. *If each member of an equation is multiplied by the same number, the result is an equation.*

If an equation is in the form $\frac{x}{2} = 6$, it can be solved by multiplying each member by 2, giving $x = 12$. Axiom III states that if this multiplication is performed, the result is true.

Axiom IV. *If each member of an equation is divided by the same number (not zero), the result is an equation.*

If an equation is in a form like $3x = 12$, it can be solved by dividing each member by the coefficient of x. Thus, dividing each member of $3x = 12$ by 3, we get $x = 4$. Axiom IV states that if this division is performed, the result is an equality.

If all terms containing the unknown letter are in one member and all numerical terms in the other, the like terms may be united and the equation solved.

Thus $5x - 2x + x = 8 + 15 - 3$ becomes, when like terms are united, $4x = 20$, and dividing each member by 4, we obtain $x = 5$.

Usually, numerical terms, as well as terms containing the unknown letter, will be found in each member of an equation, as in $5x + 3 = 2x + 18$. By the use of one or more of the preceding axioms it is always possible to change the form of such equations until they are similar to the equation $3x = 15$, which, as we have seen, can easily be solved.

In changing the form of an equation, by an application of the foregoing axioms, it is important to note that we do not change the value of the unknown. We merely discover the value which it really had all the time. In fact, the chief significance of the foregoing axioms lies in their application to any kind of equation, whether it involves numbers, letters standing for known numbers or for unknown numbers, or all of these together.

ORAL EXERCISES

Find the value of the unknown in each of the following equations. State the axiom or axioms used in solving each:

1. $2x = 4$.
2. $3x = 12$.
3. $5y = 15$.
4. $2n = 16$.
5. $7x = 35$.
6. $x + 1 = 2$.
7. $k + 3 = 5$.
8. $p + 5 = 11$.
9. $x + 2 = 17$.
10. $y + 9 = 16$.
11. $y - 1 = 1$.
12. $x - 2 = 3$.
13. $z - 4 = 5$.
14. $x - 5 = 1$.
15. $h - 7 = 3$.
16. $\frac{x}{2} = 1$.
17. $\frac{n}{2} = 3$.
18. $\frac{x}{3} = 3$.
19. $\frac{y}{5} = 4$.
20. $\frac{x}{7} = 2$.
21. $x + 4 = 7$.
22. $n - 2 = 6$.
23. $5x = 30$.
24. $y + 3 = 11$.
25. $y - 6 = 3$.
26. $\frac{x}{6} = 4$.
27. $\frac{n}{4} = 4$.
28. $7x = 7$.
29. $\frac{x}{7} = 7$.
30. $2x + 1 = 3$.
31. $2q + 5 = 11$.
32. $5n - 9 = 6$.
33. $3x = x + 4$.
34. $2x + 1 = 2$.

EXAMPLE

Solve $5x - 8 = 2x + 19$.

Solution.	$5x - 8 = 2x + 19$.	
Subtracting $2x$ from each member,	$3x - 8 = 19$.	Ax. II
Adding 8 to each member,	$3x = 27$.	Ax. I
Dividing each member by 3,	$x = 9$.	Ax. IV

This method of solution illustrates the following

Rule. *By the applications of Axioms I and II change the equation so that all of the terms containing the unknown number are in one (usually the left) member of the equation, and all other terms in the other member.*

Combine the terms in each member.

Divide each member by the coefficient of the unknown.

Checking the solution of an equation is often called *testing* or *verifying* the result. For this we have the

Rule. *Substitute the value of the unknown obtained from the solution in place of the letter which represents the unknown in the original equation. Then simplify the result until the two members are seen to be identical.*

Check. $5x - 8 = 2x + 19.$

Substituting 9 for x, $5 \cdot 9 - 8 = 2 \cdot 9 + 19.$

Simplifying, $45 - 8 = 18 + 19,$

or $37 = 37.$

EXERCISES

Find the value of the unknown in each of the following equations and verify results:

1. $x + 5 = 7.$

2. $y - 9 = 6.$

3. $4x - 6 = 14.$

4. $4x + 2 = 3x - 4.$

5. $5x - 2 = 4x + 5.$

6. $5p = 3p + 8.$

7. $4x + 2 = 2x.$

8. $3n = n + 3.$

9. $4x + 2 = x + 5.$

10. $5x + 6 = 3x + 10.$

11. $4x - 7 = 2x + 3.$

12. $3k - 5 = 7 - k.$

13. $6 - a = 9 - 4a.$

14. $2x = 1 - x.$

15. $5 + 3y = y + 4.$

16. $2x + 8 = 5x - 1.$

HINT. Subtract $2x$ from each member, etc.

17. $5 + 2q = 5q - 1.$

18. $2 - 3n = n + 1.$

19. $4x - 2 = 7 + x.$

20. $4x + 3 - x + 5 = x - 10.$

21. $3 + 2z - 1 = 12 - 3z.$

22. $2x + 3x + x = x + 9 + 2x.$

23. $4y - 3 + 3y - 4 = 6y - 8.$

24. $36 - 8x + 4 - 9x = 2x + 10.$

25. $3x + 12x + 17 - x + 4x = 107.$

26. $4k - 3 = 5k - 16 - 43k - 71.$

ORAL EXERCISES

1. A rope is 10 feet long. How long will it be if 3 feet are cut off one end?

2. If 3 feet are cut off one end of a rope x feet long, how much remains?

3. A river is d feet deep at a ford, and 8 feet deeper under a bridge. How deep is it under the bridge?

4. A 25-gallon tank is full of gasoline. If g gallons are pumped out, how many gallons are left?

5. A 50-gallon tank is full of gasoline. If gasoline is pumped out until there are only x gallons left in the tank, how many gallons are removed?

6 If the sum of two numbers is 100, and one of them is n, what is the other?

7. A pole y feet long is cut into two pieces. If one piece is f feet long, how long is the other?

8. If the sum of two numbers is s and one of them is 10, what is the other?

9. A certain chimney is three times as high as a neighboring telegraph pole. If the pole is h feet high, how high is the chimney?

10. A rectangle is three times as long as it is wide. If it is x feet wide, how long is it? What is its perimeter?

11. There are twice as many boys in a certain class as there are girls. If there are n girls, how many boys are there? How many pupils all together? How many people in the room, including the teacher?

12. The roof of a certain building is five times as far from the ground as the bottom of the third-story windows. If the windows are y feet from the ground, how high is the roof? How long would a wire have to be to reach from the roof to the ground and back to a third-story window? How long would it have to be to reach from the roof to the bottom of the window direct?

13. John's father is twice as tall as John, and 6 inches taller than John's mother. If John is x inches tall, how tall is his father? his mother?

Even integers are those exactly divisible by 2. Odd integers are those not exactly divisible by 2.

Consecutive integers are integers arranged in the natural order, like 6, 7, 8, 9, 10, etc.

Consecutive odd integers are odd integers arranged in the natural order, like 5, 7, 9, 11, 13, etc.

Consecutive even integers are even integers arranged in the natural order, like 6, 8, 10, 12, 14, etc.

14. What is the difference between any two consecutive integers? between any two consecutive odd integers? between any two consecutive even integers?

15. If n is an integer, what is the next consecutive integer? If x is an integer, what are the next two consecutive integers?

16. If x is an odd integer, what is the next consecutive odd integer? the next two consecutive even integers?

17. If e is an even integer, what is the next consecutive even integer? the next three consecutive odd integers?

18. If x is an odd integer, is $x+1$ odd or even? If x is even, what of $x+1$?

19. If x is an odd integer, what of $x+3$? $x+6$? If y is even, what of $y+2$? $y+3$?

20. If x is an integer, is $2x$ odd or even? Is $2x+1$ odd or even?

21. Is the sum of two consecutive integers always odd?

HINT. Express the integers algebraically, add them, and divide by 2.

22. Is the sum of three consecutive integers always even?

23. By what number is the sum of three consecutive integers always divisible?

Express the following statements as equations:

24. The sum of 5 and x is 8.

25. The sum of x and 4 is 9.

26. x is equal to 5 diminished by 1.

27. x is equal to 10 diminished by 3.

28. x is 4 more than 6.

29. Three times x is 15.

30. Four times a diminished by 2 equals 18.

31. x is 6 less than 9.

32. Three times y is greater by 4 than 10.

33. One half of x increased by 5 equals 9.

34. Two thirds of x increased by 6 equals 14.

35. 8 added to x gives the same result as x taken from 22.

36. $2x$ diminished by 9 is equal to x increased by 14.

37. If 6 is taken from three times x, the result is the same as when x is taken from 14.

28. Solution of problems. In the solution of problems in simple equations the following steps are necessary:

1. *Read the problem carefully and find the statement which will later be expressed by the equation.*
2. *Represent the unknown number by a letter.*
3. *Express the conditions stated in the problem as an equation involving this letter.*
4. *Solve the equation.*
5. *Check by substituting in the problem the value found for the unknown.*

In the preceding sentence the words "in the problem" are of importance, for substituting the value found in the equation would not detect any errors made in translating the words of the problem into the equation.

The foregoing directions for the solution of the various problems leading to simple equations are as definite as can be given. The student will obtain much aid from the study of the typical solutions which occur from time to time. Then one or more careful readings of each problem, a little fixing of the attention upon it, and an application of common sense will insure progress.

EXAMPLE

A man wishes to give 75 cents to his two children so that the older receives 15 cents more than the younger. How much shall he give to each?

Solution. The amount the older child receives + the amount the younger child receives = 75 cents. Let x denote the number of cents the younger child receives. Then $x + 15$ denotes the number of cents the older child receives.

By the conditions of the problem,

$$x + 15 + x = 75.$$

$$2x + 15 = 75.$$

$$2x = 60.$$

$$x = 30.$$

$$x + 15 = 45.$$

Hence the younger child receives 30 cents, and the older receives 45 cents.

Check. $75 = 45 + 30$, and $45 - 30 = 15$.

PROBLEMS

1. One boy sells 5 more newspapers than another. Together they sell 67 papers. How many does each boy sell.

2. It is desired to divide a class of 51 students into two groups, one of which shall contain 3 more than the other. How many will each group contain?

3. One number is five times another, and their sum is 30. Find both numbers.

4. The sum of two numbers is 72, and the greater is three times the less. Find both numbers.

5. One number is five times another, and their difference is 52. Find both numbers.

6. A freight train contains 72 cars. It is desired to divide it into two trains, of which one shall contain twice as many cars as the other. How many cars will there be in each train?

7. The perimeter of a certain square is 24 feet. Find the length of each side.

8. A rectangle is three times as long as it is wide, and its perimeter is 64 feet. Find its length and its width.

9. The perimeter of a certain rectangle is 78 feet. Its length is five times its breadth. Find its dimensions.

10. A rectangle is 12 feet longer than it is wide, and its perimeter is 96 feet. Find its length and its width.

11. It is desired to cut a log into two pieces so that one piece shall be 8 feet longer than the other. If the log is 36 feet long, how long will each piece be?

12. A stick 120 inches long is to be cut into two pieces, one of which is to be 14 inches more than three times the length of the other. How long is each piece?

13. The sum of three numbers is 22. The second is twice the first, and the third is four times the second. Find each number.

HINT. In solving a problem involving three or more unknown numbers the student should read the statement carefully and decide which of the numbers to be found he should indicate by x. He should select for this purpose the number in terms of which the other numbers are most easily expressed.

14. The first of three numbers is twice the third, and the second is four times the third. The sum of the three numbers is 63. Find each number.

15. It is desired to divide a company of 95 soldiers into three squads of which the second contains 5 more men than the first, and the third contains 10 more than the second. How many men will there be in each squad?

16. It is desired to divide a train of 62 cars into two sections, one of which is to contain 2 more than twice as many cars as the other. How many cars will there be in each section?

17. Find two consecutive numbers whose sum is 45.

18. Find two consecutive odd numbers whose sum is 36.

19. Find three consecutive numbers whose sum is 72.

20. Find three consecutive even numbers whose sum is 48.

21. Find five consecutive odd numbers whose sum is 85.

22. It is desired to cut a stick 36 inches long into three pieces whose lengths are consecutive integers. How long will the pieces be?

BIOGRAPHICAL NOTE. *John Wallis.* Among those who introduced and helped to standardize the modern algebraic symbolism was John Wallis, an Englishman. He was the son of a clergyman, and, like most scholars of his day, did not confine his interest to any one subject. He was at various times an instructor in Latin, Greek, and Hebrew, and for many years was professor of mathematics at Oxford. He also invented a method of teaching deaf mutes to talk.

During the wars between Charles I and Cromwell, Wallis's sympathies were with Cromwell, and he was of great service in reading royalist dispatches written in cipher. In fact, he was one of the most famous cryptologists of his day.

Wallis did not become interested in mathematics till the age of thirty-one, but devoted himself to the subject for the rest of his life. One of the earliest and most important books on algebra ever written in English was his treatise published in 1685. It contains a brief historical sketch of the subject which is unfortunately not entirely accurate, but his treatment of the theory and practice of arithmetic and algebra has made the book a standard work for reference ever since.

JOHANNES WALLIS S.T.P. GEOMETRIÆ PROF. SAVILIANUS OXON. REG. MA.TI A SACRIS. REGALIS SOCIET. LOND. SODALIS
D. Loggan ad vivum delin.

JOHN WALLIS

CHAPTER V

SUBTRACTION

29. Subtraction of monomials. The principles stated on page 25 apply to the subtraction of monomials as well as of the positive and negative numbers there used. Hence for subtracting one monomial from another we have the

Rule. *Change the sign of the subtrahend; then find the algebraic sum of this result and the minuend.*

As soon as possible the student should learn to change the sign of the subtrahend *mentally.*

EXAMPLES

1. From $+8a$ take $+5a$.

Solution. $+8a$ minus $+5a = 8a - 5a = 3a$.

2. From $6ax$ take $-2ax$.

Solution. $6ax$ minus $-2ax = 6ax + 2ax = 8ax$.

3. Subtract $7a^2b$ from $-10a^2b$.

Solution. $-10a^2b$ minus $7a^2b = -10a^2b - 7a^2b = -17a^2b$.

The difference of two dissimilar monomials cannot be written as a single term, but is expressed by a binomial, as follows:

4. Subtract $+a$ from $+b$.

Solution. $+b$ minus $+a = b - a$.

5. Subtract $-4b$ from $3c$.

Solution. $3c$ minus $-4b = 3c + 4b$.

6. Subtract $4xy$ from $-5x^2z$.

Solution. $-5x^2z$ minus $4xy = -5x^2z - 4xy$.

ORAL EXERCISES

In each of the following, subtract the second term from the first:

1. $8a$, $3a$.
2. $9a$, $6a$.
3. $11a$, $14a$.
4. $7a$, $12a$.
5. $4a$, $-2a$.
6. $6a$, $-5a$.
7. $5c$, $-7c$.
8. $4c$, $-9c$.
9. $-3a$, $6a$.
10. $-5a$, $8a$.
11. $-7a$, $-10a$.
12. $-11x$, $6x$.
13. $-6a$, $-4a$.
14. $-8a$, $-6a$.
15. $-4a$, $-7a$.
16. $-3a$, $-8a$.
17. $-9x$, $-12x$.
18. $12x$, $-5x$.
19. $-4ac$, $-4ac$.
20. $-4ac$, $4ac$.
21. $4ac$, $-4ac$.

Subtract the first monomial from the second, and also the second monomial from the first, in each of the following:

22. $2x$, $5x$.
23. $4x$, $7x$.
24. $-2x$, $-3x$.
25. $-5x$, $-3x$.
26. $-x$, $4x$.
27. $-x$, $-3x$.
28. $-3c$, $5c$.
29. $-ac$, $-5ac$.
30. $8a^2c$, $-11a^2c$.
31. $5x^2y^2$, $-5x^2y^2$.
32. $-4ab$, 0.
33. x, y.
34. c, $3x$.
35. x, $-5y$.
36. $-4a$, $6b$.
37. $-2a$, $-5b$.
38. $-2a$, $-2a$.
39. $5x$, $-5x$.

30. Subtraction of polynomials. For the subtraction of polynomials we have the

Rule. *Write the subtrahend under the minuend so that similar terms are in the same column.*

Then, changing the sign of each term of the subtrahend mentally, apply the rule on page 49 to each column.

Check. *In algebra, as in arithmetic, work in subtraction is checked by use of the relation*

difference + subtrahend = minuend

EXAMPLE

Subtract $5x - 2y - 7z + 2$ from $3x + 8y - 5z$, and check the result.

Solution.

$$\begin{array}{rrrrl} 3x + & 8y - & 5z & & = \text{minuend} \\ 5x - & 2y - & 7z + & 2 & = \text{subtrahend} \\ \hline -2x + & 10y + & 2z - & 2 & = \text{difference} \end{array}$$

Check.

Difference + subtrahend $\quad 3x + 8y - 5z = \text{minuend}$

EXERCISES

In Exercises 1–12, subtract the first number from the second:

1. $a + 2, a + 3.$

2. $a - 4, a - 6.$

3. $2a - 3, 3a + 7.$

4. $3a + 7, 9 - 4a.$

5. $2x - 5, 5 - 2x.$

6. $6x - 3, 6 + 3x.$

7. $9x + 4, 4x - 9.$

8. $4a, 4a - 2.$

9. $2ab - 5c, 5c.$

10. $3xy - a, 5xy.$

11. $0, x + 4.$

12. $2x - 5, 0.$

In Exercises 13–26, subtract the first polynomial from the second and check the work:

13. $x - 2y - 3z, 2x - 2y + z.$

14. $4x - 8y + 2z, 4x - 5y - 3z.$

15. $3a - 2b, 4a + b + 2c.$

16. $5a - 4b + 3c, 3a - 5b.$

17. $3x^2 - x - 5, 5x^2 + 8x - 2.$

18. $7x^2 - 4x + 11, 8x - 5x^2 - 13.$

19. $a - x + y, b - x - y.$

20. $3a - 2b + c + 6, 4a - b + 5c.$

21. $2a + 2b - 2c - 4, a - 3b.$

22. $2a - 3x + 4c, 4x - 3a - 3c - 11.$

23. $a + b - c, c - a - b.$

24. $2a^2 - 3ab + b^2, 3b^2 - 7a^2 + 8ab.$

25. $3x^3 - 3x^2y - 3xy^2$, $4x^2y - 5xy^2 - y^3$.

26. $a^3 - c^3 - 3a^2c + 3ac^2$, $5a^2c + 4c^3 + 3a^3 + 2ac^2$.

In Exercises 27–35, find the expression which added to the first will give the second:

27. $x - 2y + z$, $2x + 5y - 3z$.

28. $7x - 9y - 3z$, $5x - 2y - 4z$.

29. $5a - 4b + 6c$, $6a - 3b$.

30. $a - b + 2c$, 5.

31. $ax^2 + bx + c$, $2bx - c$.

32. $3ab + c$, $3xy - z$.

33. $2x - 4y - z$, 0.

34. $a^2 - 2ac + 3c^2$, $2c^2 - 3ac + 4a^2$.

35. $2x^2y - 3xy^2$, $y^2x + yx^2 - z$.

In Exercises 36–42, find the expression which subtracted from the first will give the second:

36. $x - 2y - z$, $3x - 2y - z$.

37. $x^2 - 7x - 10$, $14x - 8 + 3x^2$.

38. $x - y - z$, $5x + 3y - 8z$.

39. $4 - 8x^2$, $x^2 + 5x - 6$.

40. $a^4 + a^2c^2 + c^4$, $2c^4 - 3a^2c^2 - 4a^4$.

41. $3a + 5b - c$, 0.

42. $4a + 6b - 8y$, $a - 12$.

43. Subtract the sum of $a^2 + 2ab - 1$ and $a^2 + 12ab - 20$ from $a^2 + 13ab - 30$.

44. Subtract the sum of $a - 3b + c$ and $4a + 5b - 6c + 4$ from $a - b + c - x$.

45. From the sum of $5x + 3x^2y - 15xy^2$ and $-6x - 12xy^2 + 7yx^2$ subtract $11x - 5x^2y + 7y^2x$.

46. From the sum of $4abc^2 - 3ab^2c + 2a^2bc$ and $6abc^2 - 5ab^2c - 4a^2b^2c$ subtract $2abc^2 - 3a^2bc + 7ab^2c$.

47. From the sum of $3x - 4xy - 2z$ and $7xy - 4z - 3x$ take the sum of $5z - 2xy - a^2bc$ and $9x - 6a^2bc - z$.

Find the algebraic sum of:

48. $(4x - 3y + 6) + (3x + 5y - 10)$.

49. $(7c + 5d - e) - (4c + 5d - 9e)$.

50. $(x^2 + 2x + 5) + (2x^2 + x - 10) - (x^2 - 5x + 3)$.

51. $(x + 3y - 2z) + (4x - 5y + 3z) - (3x - 2y - 6z)$.

52. $(5x + 3y - z) + (4y + 7z) - (x - y + 3z)$.

53. $4x - 3y + 7 - (2x - 5y - 4) + (4x - 8)$.

54. $3c - 5d - e - (5c + 6d + 11e) - (5c + 4e)$.

55. $3a + 3b - 4c - (-3b - 3c - 4) - (4a + x - 8c)$.

56. $x^3 - c^3 + 3cx^2 - 3c^2x - (x^3 + c^3 - 2c^2x - 4cx^2)$.

CHAPTER VI

IDENTITIES AND EQUATIONS OF CONDITION

31. Kinds of equations. Equations are of two kinds: **identities** and **equations of condition.**

An **identity** is an equation in which, if the indicated operations are performed, the two members become precisely alike, term for term.

Thus, $4 \cdot 5 + 3 \cdot 4 = 8 \cdot 5 - \frac{4 \cdot 6}{3}$ is an identity, for, if we perform the indicated operations, it becomes $20 + 12 = 40 - 8$, or $32 = 32$.

Similarly, $2a + 3b - 4 = 3a - 2b + (5b - a - 4)$ is an identity, for, if we perform the indicated addition in the second member, the equation becomes $2a + 3b - 4 = 2a + 3b - 4$, in which the two members are alike, term for term.

An identity which involves letters is true for *any* numerical values of the letters in it.

Thus the literal identity $(a + 3)^2 = a^2 + 6a + 9$ becomes, when $a = 5$, $(5 + 3)^2 = 5^2 + 6 \cdot 5 + 9$, or $8^2 = 25 + 30 + 9$, or $64 = 64$. If a is zero, the identity becomes $(0 + 3)^2 = 0 + 6 \cdot 0 + 9$, or $9 = 9$.

An equation in one unknown which is true only for certain values of the unknown is an **equation of condition,** which for brevity we shall usually refer to simply as an equation.

The statement $4x = x + 12$ is true only when $x = 4$. If 4 is substituted for x, the equation becomes the identity $4 \cdot 4 = 4 + 12$, or $16 = 16$. Clearly the statement is false if 0, or 3, or any value other than 4 is put for x; it is true on condition that x be 4, and on no other.

Similarly, $x^2 - 5x + 6 = 0$ is true when $x = 2$ or when $x = 3$. In the first case $x^2 - 5x + 6 = 0$ becomes $2^2 - 5 \cdot 2 + 6 = 0$, or $4 - 10 + 6 = 0$, or $0 = 0$. In the second case we obtain $3^2 - 5 \cdot 3 + 6 = 0$, or $9 - 15 + 6 = 0$, or $0 = 0$. Plainly the statement obtained is false when -2 is put for x, for then it becomes $(-2)^2 - 5(-2) + 6 = 0$, or $4 + 10 + 6 = 0$, or $20 = 0$. Similarly, any value other than 2 or 3, when put for x, gives a relation between numbers which is not true.

Every equation of condition may be regarded as asking a question. Thus the equation $3x + 2 = 15$ asks, "What number when multiplied by 3 and the product increased by 2 gives 15 as the result?"

The equations used in solving the problems on pages 46–48 are equations of condition. The conditions there expressed in ordinary language in the problems were translated into the algebraic language of equations.

Instead of the equality sign, the sign $\equiv$ (read *is identical with*, or *is identically equal to*) is sometimes used for emphasis if the expression is an identity.

Thus $3a = 2a + a$ may be written $3a \equiv 2a + a$.

32. Root of an equation. A number or literal expression which being substituted for the unknown letter in an equation reduces it to an identity is said to **satisfy** the equation.

Thus it has been shown that 4 satisfies the equation $4x = x + 12$, and, similarly, the literal expression $3a$ satisfies the equation $x - 5 = 3a - 5$.

A number or number symbol is called a root of an equation if it satisfies the equation.

The process of checking the solution is really finding out whether the result obtained is a root of the equation or not. It should be particularly noted that after the root has been substituted in place of the unknown, the equation of condition becomes an identity.

ORAL EXERCISES

In Exercises 1–10, state which expressions are identities and which are equations of condition:

1. $3 + 7 = 13 - 3.$

2. $2x = 8.$

3. $3a - 5 = 2a.$

4. $x + 1 = 1 + x.$

5. $x - 2 = 2x - 3.$

6. $2x = 4x - 3x + x.$

7. $x + 7 = 4 + x + 3.$

8. $3x - 1 + x + 1 - 4x = 0.$

9. $x + 4 - 3x + 2 = 6x - 8.$

10. $3x - 3 = 4x - 2 - x - 1.$

In Exercises 11–20, select from the numbers at the right of each equation those which satisfy that equation:

11. $4x - 12 = 0.$ $1, 2, 3.$

12. $2x = x + 2.$ $1, 2, 4.$

13. $x + 3 = 4x - 6.$ $0, -1, 3.$

14. $3x - 4 = 2x + 1.$ $-2, 1, 5.$

15. $x + 6 = 3x - 10.$ $8, 5, -4.$

16. $x + 1 - 2x + 3 = 0.$ $7, 8, -3.$

17. $x^2 + 3x + 2 = 0.$ $-1, 1, 0.$

18. $x^2 - 16 = 6x.$ $8, -2, 3.$

19. $4x^2 + 4x = 3.$ $2, \frac{1}{2}, 3.$

20. $5x^2 = 3x.$ $1, 2, 0.$

Is the number at the right of each of the following equations a root of that equation?

21. $3x - 24 = 0.$ $8.$

22. $4x - 15 = x.$ $3.$

23. $5x - 12 = 2x.$ $2.$

24. $x^2 + 6x - 3 = 0.$ $4.$

25. $x^2 - 16 = 4x + 5.$ $7.$

26. $3x + 7 = 5x - 1.$ $4.$

27. $y^2 + 5y + 4 = 0.$ $2.$

28. $z + 3 = 2z - 1.$ $4.$

33. Transposition. In solving the equation $5x - 4 = 18$ we add 4 to each member. If we indicate this addition, the equation becomes

$$5x - 4 + 4 = 18 + 4.$$

Now in the first member $-4+4=0$, so that these two numbers may be omitted, and the equation becomes

$$5x=18+4.$$

Upon comparing this with the original equation, it is seen that -4 has vanished from the first member of the original equation, and $+4$ has appeared in the second member of the new equation.

Again, if we subtract $3y$ from both members of the equation

$$6y=3y+12,$$

we get the equation $6y-3y=12$,

which differs from the original equation only in having $-3y$ in the first member instead of $+3y$ in the second.

It thus appears that a term may be omitted from one member of an equation, provided the same term with its sign changed from + to − or from − to + is written in the other member. This process is called **transposition.**

Hereafter, in order to simplify an equation, instead of subtracting a number from both members or adding a number to both members, as illustrated in the example on page 41, the student should use transposition, as it is usually more rapid and convenient. He should, however, always remember that the transposition of a term is really the subtraction of that term from each member of the equation.

Like terms in the same member of an equation should be combined before transposing any term.

NOTE. Our word *algebra* is derived from the Arabic word for *transposition.* The process by which one passes from the equation $px-q=x^2$ to the equation $px=x^2+q$ was known as *al-jebr.* This is the first word in the title of an Arabic book on algebra which was translated into Latin. For some reason only this part of the title remained, and by the early part of the seventeenth century *al-jebr*, or algebra, was the common name given to the whole subject.

ORAL EXERCISES

State the term or terms which must be added to both members of the following equations in order to transpose the underscored terms. What does each equation become after transposition?

1. $x^2 - \underline{5} = 4.$
2. $x + \underline{12} = 4.$
3. $x - 3 = \underline{2}.$
4. $x^2 + 6x = \underline{5x} - 3.$
5. $x^2 - 4x = \underline{5}.$
6. $2x^2 + 3x + 1 = \underline{2x - 4}.$
7. $2x + \underline{3} + 3x + \underline{5} = \underline{4x} + 6.$
8. $5x - \underline{2} - 3x + \underline{6} = 7 - \underline{x}.$
9. $x^2 + 4x = -\underline{3}.$
10. $3x - 2 = -\underline{x^2}.$
11. $5x^2 - 4x = \underline{3x} - \underline{1}.$
12. $8x - 2x^2 = \underline{4} - \underline{4x^2}.$

EXAMPLE

Solve the equation $5x - 7 + 2x + 6 = 11x - 5 - 6x + 12.$

Solution. $5x - 7 + 2x + 6 = 11x - 5 - 6x + 12.$

Combining like terms, $7x - 1 = 5x + 7.$

Transposing, $7x - 5x = 7 + 1.$

Combining like terms, $2x = 8.$

Dividing by 2, $x = 4.$

Check. $5x - 7 + 2x + 6 = 11x - 5 - 6x + 12.$

Substituting 4 for x, $20 - 7 + 8 + 6 = 44 - 5 - 24 + 12.$

Combining, $27 = 27.$

EXERCISES

Solve the following equations, and check results:

1. $4x - 1 = 3x + 3.$
2. $4x + 5 = x - 10.$
3. $-6y - 2 = 1 - 9y.$
4. $6x + 3 - 2x = 27.$
5. $4y + 3 = 2 + 5y.$
6. $4k + 1 - 2k = 4.$
7. $16 + 4x - 4 = x + 12.$
8. $4x - 15 = 15 - x - 5.$
9. $5n + 11 - 2n = 6 - 4.$
10. $2x - 1 = 29 + 7x.$

11. $3k + 9 + 5k - 33 = 0.$

12. $3h - 20 + 8h - 24 = 0.$

13. $6x - 13 + 2x + 3 = 0.$

14. $x + 2 - 5x = -9x + 12.$

15. $4x + 3 = 15 + 2x + 6.$

16. $3y + 5 + y + 3 = 0.$

17. $9 - 8h + 2 = 3 - 4h.$

18. $3 - 5x + 2 = 5 + 7x.$

19. $8x - 3x = 15x + 4 - 13x.$

20. $8 + 7y - 13 = y - 27 - 5y.$

21. $3x - 15 - 10x - 9 + 16x - 21 = 0.$

22. $18 + 5x - 6 - 2x + 1 + 3x - 25 = 0.$

23. $0 = 18 - 4x + 27 + 9x - 3 + 16x.$

24. $5n - 8 + 4n + 5 = 7n - 3 - 2n + 5.$

ORAL EXERCISES

Represent a number:

1. Greater by 4 than x.

2. Greater by a than x.

3. Less by 5 than n.

4. Less by b than n.

5. Three times y.

6. Four greater than $a + b$.

7. x greater than $a + b$.

8. Seven less than $x - 2y$.

9. a less than $2c - b$.

10. c less than three times x.

11. a greater than four times n.

12. Three less than five times y.

13. Eight greater than three times n.

14. One part of 10 is 4. What is the other part?

15. One part of x is 5. What is the other part?

16. One part of 8 is y. What is the other part?

17. One part of n is p. What is the other part?

18. One part of x is a. What is the other part?

19. One part of $x + y$ is z. What is the other part?

20. The sum of two numbers is 12. If one of them is 5, what is the other?

21. The sum of two numbers is 12. If one of them is p, what is the other?

22. The difference of two numbers is 8. If the greater is 25, what is the other?

23. The difference of two numbers is 8. If the greater is x, what is the other?

24. The sum of two numbers is 25. If one of them is a, what is the other?

25. The sum of two numbers is s. If one of them is 9, what is the other?

26. The sum of two numbers is a. If one of them is x, what is the other?

27. The difference of two numbers is 6. The less number is 4. What is the other?

28. The difference of two numbers is d. If the less of them is 4, what is the other?

29. The difference of two numbers is d. If the less of them is n, what is the other?

30. What is the excess of 19 over 6? 9 over n?

31. By how much does 24 exceed 16? 24 exceed x? y exceed 8? $a + b$ exceed 12?

32. How much greater is 45 than 18? than 33? than a? How much greater is x than y?

33. How much less is 15 than 32? than a? How much less is x than y?

34. By how much does $a + 6$ exceed a? $a + 6$ exceed b? $4x - 3$ exceed $3x$?

34. Translation of problems into equations. In the solution of problems the writing of the equation is nothing more than translating from ordinary speech into the language of algebra. Sometimes it is possible to translate the statement of the problem, word by word, into algebraic symbols.

For example,

Four times a certain number, diminished by 6,
$4 \quad \times \quad n \quad - \quad 6$
gives the same result as the number increased by 30.
$= \quad n \quad + \quad 30$

PROBLEMS

1. What number increased by 16 is equal to 37?

Hint. $n \quad + \quad 16 \quad = \quad 37$

2. What number diminished by 23 is equal to 9?

3. To what number must 28 be added so that the result may be 15?

4. From what number must 15 be subtracted so that the result may be 24?

5. If a certain number is doubled and the result diminished by 13, the remainder is 49. What is the number?

6. If a certain number is doubled and the result increased by 16, the sum is 12. What is the number?

7. If a certain number is trebled and the result diminished by 16, the remainder is equal to the original number. What is the number?

8. Three times a certain number, less 23, equals twice the number, less 8. Find the number.

9. Five times a certain number, increased by 12, equals eight times the number, diminished by 6. Find the number.

10. Four times a certain number, less 7, equals twice the number, plus 21. Find the number.

11. What number is as much less than 90 as it is greater than 16?

12. What number exceeds 16 by twice as much as 52 exceeds the number?

13. A certain number added to 14 gives the same result as that obtained when this number is subtracted from 58. What is the number?

14. If 9 is added to twice a certain number, and 12 is subtracted from five times the number, the results are the same. Find the number.

15. Five times a certain number, plus 16, equals three times the number, plus 10. What is the number?

16. The sum of two numbers is 62, and their difference is 10. Find the numbers.

Solution. The greater number + the less number = 62.

There are two unknowns in this problem, but both can be represented in terms of the same letter, thus:

Let $n =$ the less number.

Then $n + 10 =$ the greater number, since the smaller is 10 less than the greater.

Placing these symbols in the principal statement, we have

$$n + 10 + n = 62.$$

Combining, $2n + 10 = 62.$

Transposing, $2n = 62 - 10.$

Combining, $2n = 52.$

Dividing by 2, $n = 26$, the less number,

and $n + 10 = 36$, the greater number.

Check. $36 + 26 = 62,\ 36 - 26 = 10.$

17. The sum of two numbers is 58, and their difference is 12. Find the numbers.

18. The sum of two numbers is 8, and their difference is 42. Find the numbers.

19. The sum of two numbers is 21, and the second is 5 greater than the first. Find the numbers.

20. The sum of two numbers is 85, and one exceeds the other by 41. Find the numbers.

21. The sum of three numbers is 55. The second is 5 greater than the first, and the third is 17 greater than the first. Find the numbers.

22. The sum of three numbers is 16. The second is 10 less than the first, and the third is 2 less than twice the first. What are the numbers?

23. Find three consecutive integers whose sum is 90.

24. Find three consecutive odd integers whose sum is 33.

25. A rectangle whose perimeter is 32 feet is three times as long as it is wide. Find its dimensions.

26. A rectangle whose perimeter is 102 feet is twice as long as it is wide. Find its dimensions.

27. The length of a rectangle is 7 feet more than twice the width. Its perimeter is 110 feet. What are its dimensions?

28. A lawyer can afford to spend only $40 per week for office help. He must pay the office boy $5; and he estimates that the stenographer's work is worth two thirds as much as the bookkeeper's. How much should he pay each?

29. Mr. Brown can allow his three children all together a dollar and a half per week for spending money. In order to pay his car fare to school John needs 50 cents per week more than Elizabeth, while James requires only half as much as Elizabeth. What allowance will each child receive?

30. A farmer offers his son a reward if he will shingle the roof of the barn in 6 working days. The roof contains 120 courses of shingles. The son wishes to plan the work so that he may lay each day two courses less than the previous day. How many courses must he lay the first day?

CHAPTER VII

PARENTHESES

35. Removal of parentheses. In solving exercises and problems it is often necessary to inclose several terms in a parenthesis. Sometimes it is necessary to inclose this parenthesis with other terms in a second parenthesis or even in a third. To avoid confusing the different parentheses, *brackets* [ ] and *braces* { } are also used.

The parenthesis, the brackets, and the braces are called *signs of aggregation.* For convenience, brackets and braces are often spoken of as parentheses.

Occasionally the vinculum, or bar, is used in the same way as a parenthesis. Thus $-\overline{m+2a}=-(m+2a)$.

In the solution of equations and in other algebraic work it is frequently necessary to remove all signs of aggregation. This removal, while it depends only on the principles of addition and subtraction which have already been learned, requires, nevertheless, some special study to acquire speed and accuracy.

The value of $12+(5-3)$ is the same as that of $12+5-3$, or 14. Similarly, $a+(b-c)=a+b-c$.

The plus signs preceding the parentheses in $12+(5-3)$ and $a+(b-c)$ belong to these parentheses respectively and vanish with them, whereas the plus signs *understood* before 5 and b within the parentheses are supplied when we write $12+(5-3)=12+5-3$ and $a+(b-c)=a+b-c$. In the expression $12+(-5-3)$ the sign of 5 must be retained, and we have $12+(-5-3)=12-5-3=4$.

Therefore we have the

Principle. *A parenthesis and the sign before it, if plus, may be removed from an expression without changing the signs of the terms which were inclosed by the parenthesis.*

In the expression $12-(5-3)$ the sign before the binomial shows that $(5-3)$ is to be subtracted from 12. Hence we change the signs of the terms in the subtrahend and find the sum of the resulting terms and the minuend.

Thus $12-(5-3)=12-5+3=10$. This is obviously correct, for $12-(5-3)=12-2=10$.

Similarly, $a-(b-c)$ becomes $a-b+c$ when the signs of the subtrahend, $(b-c)$, are changed and the result is added to a.

The minus signs preceding the parentheses in $12-(5-3)$ and $a-(b-c)$ vanish with the parentheses, and the plus signs understood before 5 and b within the parentheses are changed when we write $12-(5-3)=12-5+3$ and $a-(b-c)=a-b+c$.

Therefore we have the

Principle. *A parenthesis and the sign before it, if minus, may be removed from an expression, provided the sign of each term which was inclosed by the parenthesis be changed.*

These principles may also be applied to remove the parentheses used to inclose the numbers in Chapter II.

When one parenthesis incloses another, either the outer or the inner parenthesis may be removed first. It is best for the beginner to use the

Rule. *Rewrite the expression, omitting the innermost parenthesis, changing the signs of the terms which it inclosed if the sign preceding it be minus and leaving them unchanged if it be plus.*

Combine like terms that may occur within the new innermost parenthesis.

Repeat these processes until all the parentheses are removed.

EXAMPLE

Remove the parentheses from $8-(3-2a)+(4-5a)$.

Solution. $8-(3-2a)+(4-5a)=8-3+2a+4-5a=9-3a$.

The student should observe that the removal of a parenthesis is merely the performance of an *indicated* addition or subtraction. Thus $3-2a$ in the example above is a subtrahend which is to be taken from $8+4-5a$ even though it comes before $4-5a$.

The student should exercise great care in the removal of a parenthesis preceded by a minus sign. Errors in this connection are common and they persist long.

ORAL EXERCISES

Read the following expressions after removing the parentheses:

1. $8+(4+2)$.
2. $8+(4-2)$.
3. $8-(4+2)$.
4. $8-(4-2)$.
5. $x+(y+z)$.
6. $x+(y-z)$.
7. $x-(y+z)$.
8. $x-(y-z)$.
9. $a-(a-b)$.
10. $a+(-a+b)$.
11. $x+y+(x-y)$.
12. $a-b-(b-a)$.
13. $a-3-(3-a+x)$.
14. $(a-c)-(c+a)$.
15. $-(2a-b)+(2b-a)$.
16. $-(a-3c)+(2-x)$.
17. $-(3a-c)-(2c-5a)$.
18. $x-(2a-3x-2c)$.
19. $(-x+a)-(a-3x)$.
20. $-(-a+x)+(-a-5x)$.

EXERCISES

Remove the parentheses and combine like terms:

1. $14-(6-3)-5$.
2. $10+(7-4)-(9-7)$.
3. $(7-5+2)-(6-4)+12$.
4. $11a-(4a-9a)+(6a-a)$.
5. $(2b-5a)-(4a-b-7a)$.
6. $a-(b-a)+(2b-3c)$.
7. $a-b-(c-d)+(a-b)-(b-c)$.

8. $(x-y)-(2y-3x)+(x-4y)$.

9. $x-(x-y-2z)-(3z+y+4)+(x-6)$.

10. $7-[8-(3-10)]-(13-25)$.

HINT. $7-[8-3+10]-13+25$, etc.

Observe that we may first remove the *inner* parenthesis, $-(3-10)$, and also remove the last parentheses at the same time. Doing this often saves rewriting and decreases the probability of error.

11. $8-[5-(4-6)]-(6-15)$.

12. $12-(8-17)-[(4-1)-8]$.

13. $16-[3-(2-5)]+(3-5)-(8-4)$.

14. $a+[2a-(3a-2b)]+(3b-2a)$.

15. $a+[5a-(3x-2a)]-(4a-3x)$.

16. $2a-[6a-(5x-4a)]+(8a-7x)$.

17. How many parentheses may be removed the first time Exercise 18 is rewritten? Exercise 19? 20? 21? 22? 23? 24? 25?

18. $(5x-6y)-[-4x-(4z-y)-2z]$.

19. $[3x-(2y+z)]-[-(3y-2x)+5x]$.

20. $[(a+3)-(x-5)]-[a+3+(x-5)]$.

21. $7-[-6-\{-4+(9-10)\}+11]$.

22. $5a-[2a+(-3a+4b)-(a-8b)+4a]$.

23. $2x-3y-[\{3z-7x-(y-4z)-9x\}+z]$.

24. $\{4a-[2a-(8a+2b)+4]-(4b-6)\}$.

25. $-5x+[15x-\{11x-(2x-7x-4)-3x\}-22]$.

26. $(4y-7x)-\{3x-[4x-(7y-4x)-(-5y+3x)]\}$.

Sometimes it is necessary to remove some of the signs of aggregation in an expression, leaving others. In the following remove the parentheses, retaining the brackets, and simplify the results as much as possible:

27. $[(a+b)+c]$, $[(a+b)-c]$.

28. $[4x+(3z-5y)]$, $[4x-(3z-5y)]$.

29. $[(a-b)+(b-2a)]$, $[(a-b)-(b-2a)]$.

30. $[(3a-2b)+(2b-3a)]$, $[(3a-2b)-(2b-3a)]$.

31. $[(a-2b)+(3c-d)]$, $[(a-2b)-(3c-d)]$.

32. $[(4x-3)+(5y-7)]$, $[(4x-3)-(5y-7)]$.

33. $[(x^2-a^2)+(y^2-2a^2)]$, $[(x^2-a^2)-(y^2-2a^2)]$.

36. Inclosing terms in parentheses. Obviously,

$$16+9-5=16+(9-5), \text{ for each equals } 20.$$

Similarly, $a+b-c=a+(b-c)$.

That the two preceding expressions are equal may be seen by removing the parentheses according to the first Principle on page 65.

These processes are illustrations of the

Principle. *One or more terms may be inclosed in a parenthesis preceded by a plus sign, without changing the sign of any of the terms.*

The expression

$$17+8-3=17-(-8+3), \text{ for each equals } 22.$$

Similarly, $a+b-c=a-(-b+c)$,

and $a-b+c=a-(b-c)$.

That the right member in each of these cases is another form of the left may be seen by removing the parentheses according to the second Principle on page 65.

These processes are illustrations of the

Principle. *One or more terms may be inclosed in a parenthesis preceded by a minus sign, provided the sign of each term thus inclosed is changed.*

EXERCISES

In the following, inclose in a parenthesis preceded by a plus sign all the terms containing the letters x or y, and inclose in a parenthesis preceded by a minus sign all the other terms:

1. $x^2 - a^2 - 2a$.

2. $12a + x^2 - 9 - 4a^2$.

3. $y^2 - 9b^2 + 6ab - a^2$.

4. $10ab + x^2 - a^2 - 25b^2$.

5. $x^2 - b^2 - 4b^2 + y^2 - a^2 - 2xy$.

6. $-4ab + x^2 - 4b^2 + y^2 - a^2 - 2xy$.

7. $16x^2 - a^2 - 16xy - 4 + 4a + 4y^2$

8. $x^2 - b^2 - 10xy + 12ab - 36a^2 + 25y^2$.

9. $2ab - a^2 + x^2 + 2xy - b^2 + y^2$.

10. $x^2 - 16xy - a^2 + 16a + 64y^2 - 64$.

11. In Exercises 1–4, inclose the last two terms in a parenthesis preceded by a negative sign.

12. In Exercises 5–10, inclose the last three terms in a parenthesis preceded by a negative sign.

CHAPTER VIII

MULTIPLICATION

37. Product of terms containing unlike letters. The student is familiar with the fact that the factors of a product may be written in any order.

For example, $2 \cdot 4 = 4 \cdot 2.$

Similarly, $a \cdot b = b \cdot a.$

This principle is called the Commutative Law of Multiplication.

Further, $2a^2 \times 3 = 2 \times 3 \times a^2 = 6a^2,$

and $2a \times 3b = 2 \times 3 \times a \times b = 6ab.$

Similarly, $6x^2 \cdot 5y^3 = 6 \cdot 5 \cdot x^2 \cdot y^3 = 30x^2y^3.$

Also, $4ab \cdot 3z^2 = 4 \cdot 3 \cdot ab \cdot z^2 = 12abz^2.$

Up to the present we have assumed that the various operations of multiplication in any product may be performed in any order.

That is, $(3 \cdot 2)4 = 3(2 \cdot 4) = 24.$ In general terms $a(b \cdot c) = (a \cdot b)c.$ This merely tells us that a multiplied by the product of b and c is the same as the product of a and b multiplied by c. This principle is called the Associative Law of Multiplication.

Biographical Note. *Sir William Rowan Hamilton.* It is strange that of all the topics treated in this book the last to be thoroughly understood by mathematicians are those appearing in the first chapters. But in all the sciences it is often most difficult to answer the questions that at first sight seem quite obvious. Any child can ask what electricity is, but the wisest scientist cannot tell. He can only explain what electricity

SIR WILLIAM ROWAN HAMILTON

does. It is easy to ask how the earth came to be revolving around the sun with the moon revolving around it, but even the deepest students of astronomy differ in their theories of how it came to be. And so in mathematics, long after many of the more complicated processes of algebra were completely understood, the simple laws of operation of numbers were surrounded with haze. One of the men who did most to clarify the nature of these laws was Sir William Rowan Hamilton (1805–1865). He was born in Dublin, Ireland, where he lived most of his life. He was a precocious boy, and at the age of twelve was familiar with thirteen languages. He devised kinds of numbers that do not follow the same laws as those that we use in algebra, and so threw a flood of light on the nature and properties of these common numbers. He was the first to recognize the importance of the Associative Law, and called it by that name. Most of his works are very advanced in character and are difficult to read.

38. Product of terms containing like letters. By the definition of an exponent (p. 9), $a^2 = a \cdot a$, and $a^3 = a \cdot a \cdot a$.

Therefore $a^2 \times a^3 = a \cdot a \times a \cdot a \cdot a = a^5 = a^{2+3}$.

Similarly, $b \times b^3 \times b^5 = b \times b \cdot b \cdot b \times b \cdot b \cdot b \cdot b \cdot b = b^9 = b^{1+3+5}$.

In like manner $3^2 \times 3^4 \times 3^5 = 3 \cdot 3 \times 3 \cdot 3 \cdot 3 \cdot 3 \times 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^{11} = 3^{2+4+5}$.

Also, $ay^2 \times y^3 = ay^5 = ay^{2+3}$,

and $2\,ab \times 3\,a^2 = 6\,a^3b = 6\,a^{1+2}b$,

and $4\,x^2yz \times 5\,xy^3 = 20\,x^3y^4z = 20\,x^{2+1}y^{1+3}z$.

Therefore we have the

Principle. *The exponent of any letter in the product is equal to the sum of the exponents of that letter in the factors.*

This is expressed in general terms, thus:

$$n^a \times n^b = n^{a+b}.$$

The law of signs for the multiplication of positive and negative numbers, given on page 27, applies to literal terms as well.

Thus
$$+2\,a^2 \times (+3\,a^5) = +6\,a^7.$$
$$+2\,a^2 \times (-3\,a^5) = -6\,a^7.$$
$$-2\,a^2 \times (+3\,a^5) = -6\,a^7.$$
$$-2\,a^2 \times (-3\,a^5) = +6\,a^7.$$

For the multiplication of two monomials we have the

Rule. *Obeying the rule of signs for multiplication, write the product of the numerical coefficients followed by all the letters that occur in the multiplier and the multiplicand, each letter having as its exponent the sum of the exponents of that letter in the multiplier and the multiplicand.*

ORAL EXERCISES

Perform the following indicated multiplications:

1. $(3)(-6)$.
2. $(-2)(7)$.
3. $(-8)(-3)$.
4. $(-4x)(5)$.
5. $3x \cdot 5x$.
6. $x^2 \cdot x^3$.
7. $x^2 \cdot x^5$.
8. $x^4 \cdot x^5$.
9. $x^2 \cdot x^2$.
10. $x^3 \cdot x^3$.
11. $x^5 \cdot x^7$.
12. $-x^2 \cdot (x^4)$.
13. $-x^3 \cdot (-x^2)$.
14. $-x^2 \cdot (-3x^3)$.
15. $-x^2 \cdot (2x^5)$.
16. $(-2x^2)(3x^3)$.
17. $(3x^4)(-2x^5)$.
18. $(4x^2)(-5x^4)$.
19. $(-3x^3)(-4x^4)$.
20. $(-6x^2)(4x)$.
21. $(-7x)(-2x^3)$.
22. $(+2a^2)(-3a^2x)$.
23. $(-4a^3x)(+2x^4)$.
24. $(-5a^2x)(-7ax)$.
25. $(+3a^2x^3)(4a^2x^2)$.
26. $(-4x)^2$.
27. $(7)(-5a)$.
28. $(3a)(-8)$.
29. $(-2y)^2$.
30. $(-9a)(-10)$.
31. $(-3ax)^2$.
32. $(4a)(-5a)$.
33. $(6abc)^2$.
34. $(-11x)(3x)$.
35. $(7x)(-3x)$.
36. $(-a)^3$.
37. $(-2a)^3$.
38. $(3a^2x)^2$.
39. $(-2a)(-4a^2)$.
40. $(5a^4)(8a^3)$.
41. $(-5x)^3$.
42. $(a^3)(-16a)$.
43. $(-4a^5)(-6a^3)$.
44. $(+3y)^3$.
45. $(4x)(5y)(6xy)$.
46. $(3x)(-2y)^2$.
47. $(-3a^2x)^2$.
48. $(3x^2)(-y)$.
49. $(5x^2y)(-2x^3)$.
50. $(-6x^3y^2)^2$.
51. $(-x^4y)(-x^2y^4)$.
52. $(2ax^2)^3$.
53. $(5a^3)(-4a^2)(-3a)$.
54. $(-2a^2)(3a^3)^2$.
55. $(3ax)(-2a^2x)(-7ax^3)$.

39. Multiplication of a polynomial by a monomial. Clearly $2(5+3)$ is equivalent to $2 \cdot 5 + 2 \cdot 3$, each expression being equal to 16.

Similarly, $a(b+c) = ab + ac$. This principle is called the Distributive Law of Multiplication.

Therefore, for the multiplication of a polynomial by a monomial, we have the

Rule. *Multiply each term of the polynomial by the monomial and write in succession the resulting terms with their proper signs.*

$$\begin{array}{rl} \textbf{Example.} & 3x^2 - 2xy + 4y - 5a - 6 \\ & \underline{2xy \qquad\qquad\qquad\qquad\qquad\quad} \\ \textit{Product,} & 6x^3y - 4x^2y^2 + 8xy^2 - 10axy - 12xy \end{array}$$

EXERCISES

Multiply:

1. $x + 2$ by 2.

2. $x - 4$ by x.

3. $x^2 + 5$ by $2x$.

4. $3x^2 + 4$ by $2x$.

5. $x^2 - 2x$ by x^2.

6. $y^2 - 3y + 2$ by $3y$.

7. $y^3 - 5y^2 + 3y - 1$ by y^2.

8. $5x^2 - 2x - 4$ by x^3.

9. $-4x^2 + 6x - 5$ by $6x^2$.

10. $2x^2 - 3x - 2$ by $-4x^2$.

11. $x^3 - 3x^2 - 4$ by $-5x^4$.

12. $2x - 3x^2 - 2x^3$ by $-2x^3$.

13. $7xy - x + y$ by $3xy$.

14. $x^2 - 2xy + y^2$ by $-3xy$.

15. $a^4 - a^2b^2 + b^4$ by $-a^2b^2$.

16. $-a^2x^2 - 2ax + 7b^2$ by $-4abx$.

17. $7x^3 - 8x^2 + 12x - 6$ by $-4x^3$.

18. $-9a^2 - 12ax + 42x^2$ by $3ax^3$.

Perform the multiplication indicated:

19. $7(2x - 3)$.

20. $5x(x - y)$.

21. $-8(3x - 7)$.

22. $-9(-4a + b)$.

23. $-3x(2x - 7)$.

24. $6x^2(9x^3 - 4x)$.

25. $-3(x^2-2x-6)$.

26. $5xy(x^2-6x+9)$.

27. $-3x(ax-bx-3cx^2)$.

28. $(x^3-ax+a^2)(-2a^2x)$.

29. $-7ab(ax^2-bx+c)$.

30. $4x^2(-3x+7x^2-x^4)$.

40. Multiplication of polynomials. Clearly $(5+3)(7-4) = 8 \cdot 3 = 24$. The multiplication may also be performed as follows: $(5+3)\ (7-4) = 5\ (7-4) + 3\ (7-4) = 35 - 20 + 21 - 12 = 24$.

Similarly, $(2x+3)(4x-5) = 2x(4x-5) + 3(4x-5) = 8x^2 - 10x + 12x - 15$, or $8x^2+2x-15$.

In general terms $(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd$.

This gives for the multiplication of polynomials the

Rule. *Multiply the multiplicand by each term of the multiplier in turn, and add the partial products.*

Example.

$$\begin{array}{lrl}
 & 3x\ -2 & \\
 & \underline{2x\ +3} & \\
\text{Multiplying by } 2x, & 6x^2 - 4x\phantom{{}-6} & = \text{first partial product.} \\
\text{Multiplying by } +3, & \underline{+9x - 6} & = \text{second partial product.} \\
\textit{Complete product,} & 6x^2 + 5x - 6 & = \text{sum of partial products.}
\end{array}$$

EXERCISES

Multiply:

1. $x+4$ by $x+3$.

2. $2x+3$ by $x+3$.

3. $4x+7$ by $3x+2$.

4. $3x-5$ by $3x+8$.

5. $3x-2$ by $2x+3$.

6. $6-4a$ by $5a-7$.

7. $2x+y$ by $x+3y$.

8. $2x-3y$ by $3x-2y$.

9. $3x-2$ by $2x-3$.

10. $-3x+11a$ by $5x-a$.

11. $ax-bx$ by $cx+dx$.

12. $-cx+d$ by $bx-cx^2$.

13. $4x-3y^2$ by $6x+5y$.

14. x^2-5x+6 by $x-3$.

15. $3x^2-3x-7$ by $2x+4$.

16. x^2-xy+y^2 by $x+y$.

17. $a^2x^2 - 2a^2x + 4a^2$ by $ax + 2a$.

18. $3x^3 - x^2 - 5x$ by $2x^3 - 5x^2$.

19. $2x^2 - 7x + 12$ by $x^2 - 3x + 5$.

20. $x^2 - 2x - 3$ by $x^2 - 2x - 3$.

Expand:

21. $(3x^2 - 5x - 1)(2x - 3)$.

22. $(x^2 - 3x - 2)(x^2 - 2x + 3)$.

23. $(t^2 - 3t + 2)^2$.

24. $(3y - y^2 - 2)^2$.

25. $(4x - 2x^2 - 5)^2$.

26. $(3x^2 - 4x + 7)^2$.

41. Powers. A **power** of a number is the product obtained by using the number as a factor one or more times.

For example, 8, or 2^3, is the third power of 2; 81, or 3^4, is the fourth power of 3; and $32x^5$, or $(2x)^5$, is the fifth power of $2x$.

42. Arrangement. A polynomial is said to be **arranged** according to the *descending* powers of a certain letter when the exponents of that letter in successive terms decrease from left to right. Thus $2x^4 - 5x^2 - 6x + 8$ is arranged according to the descending powers of x. Again, $4 - 2y + y^2$ and $x^3 - 3x^2y + 3xy^2 - y^3$ are arranged according to the *ascending* powers of y.

Whenever it is possible to arrange the multiplier and the multiplicand in the same order with respect to the same letter it should be done, as the addition of the partial products is then much more easily performed.

43. Degree. The **degree** of a term with respect to any letter which does not appear in the denominator is determined by the exponent of that letter in the term.

Thus x, $3xy$, and $4a^2xz$ are of the first degree in x, and $3xy^2$ is of the second degree in y.

The degree of a term with respect to *two or more* letters which do not appear in the denominator is determined by the *sum of the exponents* of those letters in that term.

Thus $5x^3y$ is of the fourth degree in x and y; $4a^2bc^3x^2y$ is of the sixth degree in a, b, and c.

44. Check of multiplication. The work of multiplication can be checked by giving a small, convenient numerical value to each letter involved and finding the corresponding numerical values of the multiplier, the multiplicand, and the product. The product of the numerical values of the multiplier and the multiplicand should equal the numerical value of the product.

The least positive integer which gives a reliable check by the method outlined above is the number 2. This is true only when one letter is involved. If more than one letter is involved, the check is not certain if 2 is substituted for each letter.

The number 1 is very convenient to use in checking, but it will not check exponents, since $x^2 = x^3 = x^5 = x^{10}$, etc., if $x = 1$.

EXAMPLE

Multiply $3x^3 - 5 + x^2 - 2x$ by $x^2 - 6 - 5x$, and check result:

Solution and check. Arranging both multiplier and multiplicand in descending powers of x and multiplying, we obtain:

$$\begin{array}{lrrrrrrlrl}
 & 3x^3 & +x^2 & -2x & -5 & & & =24+4-4-5= & 19 & (\text{If } x=2)\\
 & & x^2 & -5x & -6 & & & =\ 4-10-6 & =-12 & \\
\hline
 & 3x^5+ & x^4- & 2x^3- & 5x^2 & & & & -228 & \\
 & & -15x^4- & 5x^3+ & 10x^2+ & 25x & & & & \\
 & & & -18x^3- & 6x^2+ & 12x+ & 30 & & & \\
\hline
\textit{Product,} & 3x^5- & 14x^4- & 25x^3- & x^2+ & 37x+ & 30 & & & \\
\textit{Product,} & 96 & -224 & -200 & -4 & +74 & +30 & = & -228 & (\text{If } x=2)
\end{array}$$

Since -228 is obtained in both steps of the check, the result is correct.

ORAL EXERCISES

1. Arrange $3x^2 - 8x^3 - 7 - x^4 - 2x$ in descending order.

2. Arrange $a^4 + b^4 + 6a^2b^2 - 4ab^3 - 4a^3b$ in descending order with reference to the letter b; with reference to the letter a.

3. Why should multiplier and multiplicand be arranged in the same order before attempting to multiply?

4. What is the value of a^3b^2 and of a^2b^3 if $a = b = 2$?

5. What point in checking does the preceding question bring out?

6. If 1 or 0 is correctly substituted for the letter or letters in a check of multiplication and the two results do not agree, is the multiplication necessarily incorrect?

7. Why is a check of multiplication unreliable when the number 1 is substituted for the letter or letters involved?

8. If one interchanges the multiplier and multiplicand and performs the multiplication a second time, would this give a check on the first? Explain.

EXERCISES

Multiply, and check every fifth exercise:

1. $3x^2 - 5x - 2$ by $2x - 3$.

2. $3x^2 + 2x - 7$ by $2x^2 - 5x - 3$.

3. $x^2 + 2x - 4$ by $x^2 + 2x + 4$.

4. $3a^2 - 8a - 1$ by $a^2 + 2a - 3$.

5. $2x^2 - 7x - 2$ by itself.

6. $2x^3 - 3x + 2$ by itself.

7. $a^2 - \frac{1}{2}a + \frac{1}{4}$ by $a^2 - a + 1$.

8. $x^2 - xy + y^2$ by $x^2 + xy + y^2$.

9. $3x^3 + 5x^2 - x + 2$ by $x^2 - 2x + 3$.

10. $a - 3 - a^3 - 2a^2$ by $2a - 3a^2 - 1$.

Hint. Arrange multiplier and multiplicand in descending order first.

11. $3a - a^3 + 6$ by $4a - 3a^2 - 5$.

Expand:

12. $(x^3 - x - 5)(3x^2 - 2x - 4)$.

13. $(3x - x^3 + x^2 - 6)(5 - x^2 - 3x)$.

14. $[3x - 2a - (2a - 3x)][3x - 2a + (2a - 3x)]$.

15. $(4a - 5a^2 + 7 + a^3)(3 + a^3 - a + a^2)$.

16. $(5x - 3 + 8x^3)(8 - 5x^2 + 2x^3 - 9x)$.

17. $(x^2 - 2xy + 3y^2)(x^2 + 2xy + 3y^2)$.

18. $(x^2y - y^2x)(4xy - 5x^2y)(3x^2y - 2xy^2)$.

19. $(x^2 + y^2 + z^2 - xy - xz - yz)(x + y + z)$.

20. $(a + b + c)^2$.

21. $(2a - 3b + 4c)^2$.

22. $(c + d - \frac{1}{2})^2$.

23. $(2a - 4b + 3)^2$.

24. $(a - 2b + 3c - 4d)^2$.

25. $(x + y + z)^3$.

Expand and collect like terms in:

26. $(x + 2y)^2 - (x - 2y)^2$.

27. $(x - 3)^3 + (3 - x)^3$.

28. $(x + y)^3 + (x - y)^3$.

29. $(2x - 3a)^3 - (3a - 2x)^3$.

30. $(ax - 5ay)^3 - (ax - ay)^3$.

31. $(4x - 3y)^2 - (2x + 5y)^2$.

32. $(x - 3)^3 - (2x - 1)^2$.

33. $(2x^2 - 3a)(2x^2 + 3a)^2$.

34. $(x^3 - 5)(x^3 + 4) - (x^5 + x^2)$.

35. $[a + (a^2 + 3)][a - (a^2 + 3)]$.

36. $[a - (x^2 - 3)][a + (x^2 - 3)]$.

37. $[x - a + (x^2 - 2)][(x - a) - (x^2 - 2)]$.

38. $(2x - 3)(3x^2 - 5x) - (4x^2 - 3x)(2x - 7) - (3x - 6)^2$.

CHAPTER IX

PARENTHESES IN EQUATIONS

45. Simple equations involving parentheses. In handling parentheses it is very easy to acquire careless habits, which are difficult to overcome. Accuracy in such work demands especial care in removing each parenthesis that is preceded by a minus sign.

EXAMPLE

Solve the equation $3(2x+1)-(4x-7)=16$.

Solution. $3(2x+1)-(4x-7)=16.$

Removing parentheses, $6x+3-4x+7=16.$

Combining, $2x+10=16.$

Transposing, $2x=6.$

Dividing by 2, $x=3.$

Check. $3(2\cdot 3+1)-(4\cdot 3-7)=16.$

Simplifying, $21-5=16,$

or $16=16.$

EXERCISES

Solve and check :

1. $2(x+3)=12.$
2. $5(x-1)=30.$
3. $4(x+6)=16.$
4. $2(3-x)+1=2.$
5. $5(x-3)+14=4.$
6. $5(x-7)+8x=4.$
7. $4(4x-1)+3=x.$
8. $16+2(4y-7)-12y=0.$
9. $5x-4(4-x)-11=0.$
10. $2(x+1)-3=3(x-1).$
11. $4(2x-5)+15=3(x+10).$
12. $3(x+6)+8=5(6+x).$
13. $7(x+5)=4(x+8)+3.$
14. $7(y-2)-2(3+y)=0.$

15. $9y - 3(2y - 4) = 2(5 - y) + 7.$

16. $2n - 9(2n + 4) = 2(n - 9).$

17. $7x - 12 - 2(x - 5) = x - 14.$

18. $5(3h - 1) - 7h = 3(h + 7) - 1.$

19. $(n - 4)(n + 8) = 7 - (3 - n)(n + 5).$

Solution. Expanding, $(n^2 + 4n - 32) = 7 - (15 - 2n - n^2).$

Removing parentheses, $n^2 + 4n - 32 = 7 - 15 + 2n + n^2.$

Subtracting n^2 from each member, and combining,

$$4n - 32 = -8 + 2n.$$

Transposing and combining, $2n = 24.$

Dividing by 2, $n = 12.$

Check. $(12 - 4)(12 + 8) = 7 - (3 - 12)(12 + 5).$

Simplifying, $8 \cdot 20 = 7 - (-9 \cdot 17);$

that is, $160 = 7 - (-153),$

or $160 = 160.$

20. $(x + 3)^2 - (x + 5)^2 = -40.$

21. $(x + 2)^2 - (x - 4)^2 + 48 = 0.$

22. $(2x - 4)(3x - 6) = 6x^2 + 72.$

23. $(x + 4)(x + 6) = (x + 18)(x + 13).$

24. $(h + 2)(h + 3) = (h - 5)(h - 2).$

25. $(k - 7)(5 + k) - (k - 5)(k + 7) + 5 = 0.$

ORAL EXERCISES

1. The length of a rectangle is $x - 3$ and its width is 4. What is its area? its perimeter?

2. The length of a rectangle is a and its breadth is b. What is its area? its perimeter?

3. Express as a product the area of a rectangle whose length is $3x + 2$ and whose breadth is $x - 1$. What is its perimeter?

4. Each of three sheep cost $40. What was the cost of all?

5. Each of n horses cost $200. What represents the cost of all?

6. Each of a books cost b cents. What represents the cost of all?

7. What is the total cost of x hats at a dollars each, and y hats at b dollars each?

8. What is the cost of h horses at $x + 15$ dollars each?

9. Represent the total cost of a chairs at $y + 4$ dollars each, and b chairs at $z - 2$ dollars each.

10. A is n years old. What will be his age 6 years hence? x years hence? What was his age 5 years ago? a years ago?

11. A's age in years is three times B's. If B is x years old, represent A's age (a) now, (b) 7 years hence, (c) 4 years ago, (d) the sum of their ages 4 years hence.

12. A's age is $2n - 3$ years. What will be his age 12 years from now? b years from now? What was his age 7 years ago? x years ago?

13. A and B each have d dollars. If A gives B five dollars, how much will each then have?

14. A and B each have $x + 25$ dollars. If B gives d dollars to A, how much will each then have?

15. If A has $x + 25$ dollars, and B has $2x + 8$ dollars, express as an equation each of the following statements:

(a) A has as many dollars as B.
(b) A and B together have $250.
(c) A has $15 less than B.
(d) If A gains $75 and B loses $20, they have equal amounts.

16. If A's age is n years, B's $3n + 5$ years, and C's $2n - 4$ years, express

(a) the ages of A, B, and C six years hence;
(b) the ages of A, B, and C five years ago.

Express each of the following statements as an equation:

(c) The sum of the ages of A and B six years hence will be 50 years.

10. The difference of the squares of two consecutive integers is 35. Find the integers.

11. The difference of the squares of two consecutive integers is 71. Find the integers.

12. The difference of the squares of two consecutive odd integers is 104. Find the integers.

13. The square of an integer plus the square of the next consecutive integer is 17 less than twice the square of the greater integer. Find the integers.

14. The difference of the squares of two consecutive odd integers is 48. Find the integers.

15. The product of two consecutive odd integers is 42 less than the square of the greater integer. Find the integers.

16. The product of two consecutive even integers equals 44 increased by the square of the smaller. Find the integers.

17. A's age in years is three times B's, and C is 10 years older than B. The sum of their ages is 45 years. Find the age of each.

18. A's age in years is twice B's, and C is 7 years older than A. Six years hence the sum of their ages will be 85 years. How old is each?

19. A is 10 years older than B, and C is 6 years younger than B. Six years ago the sum of their ages was 40 years. Find the age of each.

20. A is 2 years more than twice as old as B, and C is 7 years younger than A. In 6 years the sum of their ages will be 75 years. Find the age of each.

21. A is now 50 and B is 36 years old. How many years ago was A three times as old as B?

22. A is now 19 years old and B is 54. In how many years will A be exactly half as old as B?

23. A is 30 years older than B. In 20 years A will be twice as old as B. Find the age of each now.

24. A is three times as old as B. In 15 years A will be twice as old as B. Find the present age of each.

25. A's age is 8 years more than twice B's age. Sixteen years ago A was four times as old as B. Find the age of each now.

26. A square has the same area as a rectangle whose length is 8 inches greater, and whose breadth is 4 inches less, than the side of the square. Find the dimensions of each.

Solution. By the conditions of the problem,

the area of the square = the area of the rectangle.

Let s = the length of the side of the square in inches.

Then $s + 8$ = the length of the rectangle in inches,

and $s - 4$ = the breadth of the rectangle in inches.

Now the area of the square is $s \cdot s$, or s^2, square inches, and the area of the rectangle is $(s + 8)(s - 4)$, or $s^2 + 4s - 32$, square inches.

Therefore $$s^2 = s^2 + 4s - 32. \qquad (1)$$

Solving (1), $s = 8$, the length of the side of the square;

$s + 8 = 16$, the length of the rectangle,

and $s - 4 = 4$, the breadth of the rectangle.

Check. The area of the square is $8 \cdot 8 = 64$ square inches, and the area of the rectangle is also $16 \cdot 4 = 64$ square inches.

27. A square field has the same area as a rectangular field whose length is 15 rods greater, and whose breadth is 10 rods less, than the side of the square. Find the dimensions of each field.

28. A tennis court, for singles, is 3 feet shorter than three times its breadth. The distance around the court is 210 feet. Find the length and the breadth of the court.

29. A tennis court, for doubles, is 6 feet longer than twice its breadth. The perimeter of the court is 228 feet. Find the dimensions of the court.

30. The breadth of a basket-ball court is 20 feet less than its length. The perimeter of the court is 80 yards. Find the dimensions.

31. The perimeter of a rectangular athletic field is 780 feet. Its length is 5 yards less than twice its breadth. Find the dimensions.

32. In a rectangle 24 feet broad and 30 feet long a grassplot is to be laid out, surrounded by a flower bed of uniform width. It is desired that the perimeter of the grassplot be exactly one half that of the entire rectangle. How wide should the flower bed be made?

33. The value of 15 pieces of money, consisting of nickels and dimes, is 90 cents. Find the number of each.

Solution. By the conditions of the problem,

the value of the dimes + the value of the nickels = 90 cents.

Let d = the number of dimes.

Then $15 - d$ = the number of nickels.

Now $10\,d$ = the value of the dimes in cents,

and $5(15 - d)$ = the value of the nickels in cents.

Therefore $10\,d + 5(15 - d) = 90.$ (1)

Solving (1), $d = 3$, the number of dimes,

and $15 - d = 12$, the number of nickels.

Check. $3 \cdot 10 + 12 \cdot 5 = 30 + 60 = 90.$

34. The value of 35 coins, consisting of dimes and quarters, is $6.50. Find the number of each.

35. The value of 30 coins, consisting of nickels and dimes, is $2.60. Find the number of each.

36. A collection of nickels, dimes, and quarters amounts to $10.80. There are 5 more nickels than dimes, and the number of quarters is double the number of nickels and dimes together. Find the number of each.

CHAPTER X

DIVISION

47. Division of monomials. The rule for division of numerical terms was stated on page 29.

Just as $2 \div 3$ is written $\frac{2}{3}$, so $a \div b$ may be written as a fraction, $\dfrac{a}{b}\cdot$

Similarly, $$a^2 \div x^2 = \frac{a^2}{x^2},$$

and $$2a \div 3b = \frac{2a}{3b}.$$

But $$12c^2 \div 4b^2 = \frac{12c^2}{4b^2} = \frac{3c^2}{b^2}.$$

By the definition of an exponent (p. 9),

$$a^5 = a \cdot a \cdot a \cdot a \cdot a \text{ and } a^2 = a \cdot a.$$

Then $$a^5 \div a^2 = \frac{\cancel{a} \cdot \cancel{a} \cdot a \cdot a \cdot a}{\cancel{a} \cdot \cancel{a}} = a^3, \text{ or } a^{5-2}.$$

Similarly, $$2^6 \div 2^3 = \frac{\cancel{2} \times \cancel{2} \times \cancel{2} \times 2 \times 2 \times 2}{\cancel{2} \times \cancel{2} \times \cancel{2}} = 2^3, \text{ or } 2^{6-3},$$

and $$ax^3 \div x^2 = \frac{a \cdot x \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x}} = ax, \text{ or } ax^{3-2}.$$

In like manner $6by^5 \div 2y^3 = 3by^2$, or $3b \cdot y^{5-3}$.

These examples illustrate the

Principle. *The exponent of any letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.*

The foregoing principle expressed in general terms is

$$n^a \div n^b = n^{a-b}.$$

What this equation means when $b = a$ and when b is greater than a will be explained later.

From what precedes we see that $ax^2 \div x^2 = \frac{a \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x}} = a.$

Hence a letter which has the same exponent in divisor and dividend should not appear in the quotient.

The law of signs in division may be indicated as follows:

$$+ab \div (+a) = +b.$$
$$+ab \div (-a) = -b.$$
$$-ab \div (+a) = -b.$$
$$-ab \div (-a) = +b.$$

Now $-12a \div 6b = -\frac{2a}{b}$. Here the quotient is a fraction, and the minus sign indicates that the fraction is negative.

Similarly, $$9x \div (-3y) = -\frac{3x}{y},$$

and $$-24a^2y \div (-6z^3) = +\frac{4a^2y}{z^3}.$$

For the division of monomials we have the

Rule. *Divide the numerical coefficient of the dividend by the numerical coefficient of the divisor, keeping in mind the rule of signs for division.*

Write after this quotient all the letters of the dividend except those having the same exponent in divisor and dividend, giving to each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor.

If there are any letters in the divisor unlike those in the dividend, write them under the preceding result as a denominator.

ORAL EXERCISES

Perform the indicated division:

1. $-20 \div 2$.
2. $12 \div (-3)$.
3. $-36 \div (-4)$.
4. $8a^8 \div a^2$.
5. $a^5 \div a^8$.
6. $-a^{14} \div a^7$.
7. $x^6 \div (-x)$.
8. $-x^9 \div (-x^8)$.
9. $ax^8 \div (-ax)$.
10. $-a^2x^5 \div a^2x$.
11. $-a^2xc \div cx$.
12. $a^2x^8c^4 \div a^2x$.
13. $-8a^6 \div 2a^2$.
14. $9x^2 \div (-3x)$.
15. $-18x^7 \div (-6x^4)$.
16. $-35ax^8 \div 5ax^2$.
17. $12ax^3 \div (-3bx^8)$.
18. $14a^8b \div (-2ab)$.
19. $-28ay^4 \div (-7cy^8)$.
20. $-28a^8x^6 \div (-4a^8x^2)$.
21. $70x^6y^9 \div (-14x^2y^8)$.
22. $16ac^4x \div 2axc^2$.
23. $48ax^5 \div (-16bx^4)$.
24. $-36x^6y^6 \div (-6x^2y^8)$.
25. $63c^8d^5 \div (-9cd^8)$.
26. $64a^5b^7 \div (-16ab^7)$.
27. $-28a^6b^{12} \div (-7a^4b^5)$.
28. $-36a^4xb^8 \div 9a^8xb^7$.
29. $\frac{15x^2y^4}{75xy^4}$.
30. $\frac{42x^{24}y^{56}}{-6x^8y^7}$.
31. $\frac{-17a^9b^8}{34a^7b^8}$.
32. $\frac{39x^{18}y^{26}}{-13x^{18}y^{18}}$.
33. $\frac{-11a^2b^{11}c^{18}}{66ac^{12}}$.
34. $\frac{49a^7x^{14}n^{21}}{7a^2x^7n^7}$.

48. Division of a polynomial by a monomial. In multiplying a polynomial by a monomial (p. 73) we multiply each term of the polynomial by the monomial. In division of a polynomial by a monomial we reverse the steps of multiplication and divide each term of the polynomial by the monomial.

Thus $$(ax + bx) \div x = \frac{ax}{x} + \frac{bx}{x} = a + b.$$

Again, $$\frac{6x^4 - 12x^3 + 18x^2}{-3x^2} = -2x^2 + 4x - 6.$$

Therefore, for the division of a polynomial by a monomial we have the

Rule. *Divide each term of the polynomial by the monomial and write the partial quotients in succession.*

ORAL EXERCISES

Perform the indicated division:

1. $\dfrac{a^3-a^4}{a^2}$.

2. $\dfrac{a^2c-a^3}{a^2}$.

3. $\dfrac{6x^2-4x}{2x}$.

4. $\dfrac{9a-18a^3}{-3a}$.

5. $\dfrac{8x-12x^3}{-4x}$.

6. $\dfrac{6ax-10a^2x^2}{2ax}$.

7. $\dfrac{9ax^3-12a^3x^5}{-3ax^2}$.

8. $\dfrac{ac+ab}{a}$.

9. $\dfrac{25x^2y+30xy^5}{-5xy}$.

10. $\dfrac{16bx^4-36b^2x^2}{4bx^2}$.

11. $\dfrac{14x^3y^4-28x^5y^6}{7x^2y^3}$.

12. $\dfrac{4x^4y-8x^6y^2+12x^8y^4}{4x^4y}$.

13. $\dfrac{a^3cd^2-a^2c^3+a^2cd^2}{a^2c}$.

14. $\dfrac{ax^4-bx^3+cx^2}{-x^2}$.

15. $\dfrac{15a^2b^2+9a^4b^3-30a^6b^4}{-3a^2b^2}$.

16. $\dfrac{16a^4b^5-24a^5b^6-48a^6b^7}{-8a^2b^2}$.

17. $\dfrac{85xyz-51x^2yz^2+102x^3yz^3-170x^5y^5z}{+17xyz}$.

18. $\dfrac{4(x+3)+a(x+3)}{x+3}$.

19. $\dfrac{2(x+1)+3x(x+1)}{x+1}$.

20. $\dfrac{(a+x)-2(a+x)^3}{a+x}$.

21. $\dfrac{(a+b)^4-3(a+b)^3}{(a+b)^2}$.

22. $\dfrac{3x(3x+4)-4y(3x+4)}{3x+4}$.

23. $\dfrac{(a-b)c-(a-b)x}{a-b}$.

24. $\dfrac{21(x-y)^7-35(x-y)^6}{-7(x-y)^5}$.

25. $\dfrac{-5(ac^2-2d)^3+x(ac^2-2d)}{5(ac^2-2d)}$.

49. Division of one polynomial by another. The process of dividing one polynomial by another is illustrated in the following:

EXAMPLES

1. Divide $x^2 - 5x + 6$ by $x - 3$.

Solution.

$$\begin{array}{r|l} x^2-5x+6 & \underline{x-3} = \text{Divisor} \\ \underline{x^2-3x} & \underline{x-2} = \text{Quotient} \\ -2x+6 & \\ \underline{-2x+6} & \end{array}$$

Check.

$$\begin{array}{r} x\ -3 \\ \underline{x\ -2} \\ x^2-3x \\ \underline{-2x+6} \\ x^2-5x+6 \end{array}$$

2. Divide $16x + 12x^3 - 15 - 22x^2$ by $2x - 3$.

Solution.

$$\begin{array}{r|l} 12x^3-22x^2+16x-15 & \underline{2x-3} \qquad = \text{Divisor} \\ \underline{12x^3-18x^2} & \underline{6x^2-2}x+5 = \text{Quotient} \\ -\ 4x^2+16x & \\ \underline{-\ 4x^2+\ 6x} & \\ +10x-15 & \\ \underline{+10x-15} & \end{array}$$

Check.

$$\begin{array}{r} 6x^2-2x+5 \\ \underline{2x\ -3} \\ 12x^3-\ 4x^2+10x \\ \underline{-18x^2+\ 6x-15} \\ 12x^3-22x^2+16x-15 \end{array}$$

The method of dividing one polynomial by another is expressed in the

Rule. *Arrange the dividend and the divisor according to the descending (or ascending) powers of some common letter, called the letter of arrangement.*

Divide the first term of the dividend by the first term of the divisor and write the result as the first term of the quotient.

Multiply the entire divisor by the first term of the quotient, write the result under the dividend, and subtract, being careful to write the terms of the remainder in the same order as those of the divisor.

To find the second term of the quotient, divide the first term of this remainder by the first term of the divisor, and proceed as before until there is no remainder, or until the remainder is of lower degree in the letter of arrangement than the divisor.

Check for division. *The product of the divisor and the quotient should equal the dividend.*

EXERCISES

Divide the following and check Exercises 1, 2, 3, 12, 17, and 18:

1. $x^2 + 10x + 24$ by $x + 4$.
2. $x^2 - 2x - 15$ by $x + 3$.
3. $x^2 + x - 6$ by $x - 2$.
4. $3x^2 + 5x + 2$ by $x + 1$.
5. $2a^2 - 3a - 2$ by $2a + 1$.
6. $5x^2 - 22x + 8$ by $x - 4$.
7. $6x^2 + 19x - 7$ by $3x - 1$.
8. $3x^2 - ax - 2a^2$ by $x - a$.
9. $4x^2 - 8ax + 3a^2$ by $2x - 3a$.
10. $x^3 - 8x^2 + 6x + 12$ by $x - 2$.
11. $8x^3 - 12x^2 + 6x - 1$ by $2x - 1$.
12. $x^3 - 11x - 6$ by $x + 3$.
13. $x^3 - 14x - 8$ by $x - 4$.
14. $x^3 - 2x^2 - 5x + 6$ by $x - 3$.
15. $x^3 - 5x + 2$ by $x^2 + 2x - 1$.
16. $x^3 - 11x + 6$ by $x^2 + 3x - 2$.
17. $x^4 - 11x^2 + 2x + 12$ by $x^2 + 2x - 4$.
18. $x^3 + 8$ by $x + 2$.
19. $8x^3 + 1$ by $2x + 1$.
20. $27x^3 + 8a^3$ by $3x + 2a$.
21. $ax + 3a - bx - 3b$ by $x + 3$.
22. $3ax - ay - 6bx + 2by$ by $2b - a$.
23. $28ax + 9ny - 21ay - 12nx$ by $3y - 4x$.

If there is no remainder, we have seen that the process of division may be expressed as follows:

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient.}$$

If there is a remainder, the following relation holds:

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Partial quotient} + \frac{\text{Remainder}}{\text{Divisor}}.$$

This last corresponds to what is done in arithmetic in dividing 17 by 5, which is written $\frac{17}{5} = 3\frac{2}{5}$. This means that $\frac{17}{5} = 3 + \frac{2}{5}$, the plus sign being understood.

General check for division. (*a*) When the division is exact. Multiply the divisor by the quotient. The product should be the dividend.

(*b*) When there is a remainder. Multiply the divisor by the partial quotient and add the remainder to the product obtained. The result should be the dividend.

NOTE. We saw on page 1 that it is customary to represent the product of two letters by placing one after the other with no sign between them. Thus *ab* means *a* times *b*. But addition, not multiplication, is implied by placing the fraction $\frac{2}{5}$ after the number 3. This practice comes down to us from the Arabs, who denoted all additions by placing the number symbols in succession without any sign of operation. The later Greeks also had the same notation.

EXERCISES

Divide the following and check Exercises 1, 7, 11, 14, and 22:

1. $6x^2 - 13x + 6$ by $2x - 3$.
2. $25x^4 + 30x^2 - 7$ by $5x^2 + 7$.
3. $6x^2 + 11x - 35$ by $2x + 7$.
4. $19a + 12a^2 - 21$ by $4a - 3$.

HINT. Rearrange the terms in the dividend.

5. $-8 + x^3 + 4x - 2x^2$ by $x - 2$.
6. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.
7. $x^3 - 15x^2 + 65x + 63$ by $x - 7$.
8. $5x^2 + 5x - 25x^3 - 1$ by $5x^2 - 1$.
9. $2x^3 - 14x^2 + 14x + 12$ by $2x - 4$.

10. $6x^3 - 18x + 12$ by $3x - 3$.

11. $4x^3 + 16x + 12$ by $x^2 + x - 3$.

12. $6x^3 - 2x^2 + 10x + 18$ by $3x^2 - 4x + 9$.

13. $3a^3 + 28a^2 + 29a - 140$ by $3a - 5$.

14. $-37x - 6x^3 - 24 + 23x^2$ by $2x - 3$.

15. $2x^2 + 10xy - 4x - 20y$ by $2x + 10y$.

16. $4x^2 + 3cx + 8ax + 6ac$ by $4x + 3c$.

17. $53a + 8 - 53a^2 + 12a^3$ by $4a^2 - 7a - 1$.

18. $-15a^3 + 56a^2 - 99a + 70$ by $3a^2 - 7a + 10$.

19. $3x^4 + 11x^3 - 3x^2 + 17x - 4$ by $x - 3x^2 - 4$.

20. $6an + 15xy - 10ny - 9ax$ by $3x - 2n$.

21. $23a^2 + a^4 - 55a + 11a^3 - 140$ by $a^2 - 5$.

22. $25x^3 - 10x^2 + 36x + 72$ by $5x - 6$.

23. $4a^3 + 1 + a^4 + 4a + 6a^2$ by $1 + a^2 + 2a$.

24. $40x - 31x^2 + 21 + x^4 - 4x^3$ by $x^2 - 3 - 7x$.

25. $a^4 - 8a^3 + 24a^2 - 32a + 16$ by $a^2 - 4a + 4$.

26. $224x - 42x^2 + 10x^4 - 27x^3 - 419$ by $9 + 2x^2 - 5x$.

27. $ax - 6cy - 2bx + 4by + 3cx - 2ay$ by $2y - x$.

28. $x^4 + a^2x^2 + a^4$ by $x^2 - ax + a^2$.

29. $a^4 + 4b^4 + 3a^2b^2$ by $2b^2 + a^2 - ab$.

30. $x^6 - 4x^3y^3 + 8y^6$ by $x^2 + 2y^2 + 2xy$.

31. $9a^4 + 49b^4 + 29a^2b^2$ by $7b^2 + 3a^2 + 4ab$.

32. $x^3 - y^3$ by $x - y$.

33. $a^3 - 125b^3$ by $a - 5b$.

34. $x^4 - 16$ by $x + 2$.

35. $x^6 - 1$ by $x - 1$.

36. $a^6 + 343b^3$ by $a^2 + 7b$.

37. $8a^3 - 64x^9$ by $2a - 4x^3$.

38. $x^5 + y^5$ by $x + y$.

39. $x^5 + y^5$ by $x - y$.

40. $x^5 - y^5$ by $x + y$.

41. $x^5 - y^5$ by $x - y$.

42. $27x^6 + 8n^{12}$ by $3x^2 + 2n^4$.

43. $y^5 - 5y^2 - 3000$ by $y - 5$.

CHAPTER XI

EQUATIONS AND PROBLEMS

50. Equations involving literal coefficients. The most general form of an equation in one unknown is that in which the unknown occurs with literal coefficients. The simplest general form is $ax = b$, in which x is the unknown. The solution of such equations involves no new principle. It is merely necessary to perform the usual operations with letters instead of with ordinary numbers. This should cause no difficulty, since in algebra the letters are really nothing but symbols for numbers, and should be used in all operations as freely as if they were integers.

In solving equations whose coefficients involve letters the answers obtained will usually involve these same letters. Only in exceptional cases may one expect to obtain the root of such an equation in purely numerical form.

In the following exercises the unknown is represented by a letter near the end of the alphabet, as x or y, while the letters in the coefficients and known terms are taken from the beginning of the alphabet.

ORAL EXERCISES

Solve for the unknown:

1. $x - a = 0.$

2. $x - 2c = 3c.$

3. $z + 5b = 7b.$

4. $x - a = a.$

5. $x + 3a = a.$

6. $x - a = b.$

7. $3x = 6a.$

8. $5y = 10c.$

9. $7z = 21b.$

10. $ax = a.$

11. $bx = ab.$

12. $ax = a^2 + a.$

13. $2y = 2a + 4b.$

14. $3ay = 6ab.$

15. $3x = 6a + 3b.$

16. $2x - 6a = 4b.$

17. $ay - ab = 2a.$

18. $2ay + 4ab = 10ab.$

19. $(a + b)x = a + b.$

20. $(a + b)y = (a + b)^2.$

21. $(a - b)x = (a - b)(a + b).$

22. $(a + b)(a - c)x = (a + b)(a - c).$

23. $(a - b)(a + b)x = 2ab(a + b)(a - b).$

EXERCISES

Solve for the unknown, and check:

1. $3x - a = x + 5a.$

Solution. $3x - a = x + 5a.$

Transposing, $3x - x = 5a + a.$

Combining, $2x = 6a.$

Dividing by 2, $x = 3a.$

Check. $3 \cdot 3a - a = 3a + 5a,$

or $8a = 8a.$

2. $bx + b = 4b.$

3. $x + a = a + b.$

4. $y - b = a - b.$

5. $3ax + 4a = 7a.$

6. $cz + c^2 = 6c^2.$

7. $ax + ab = ac.$

8. $4bx - b^2 = 5b^2.$

9. $a^2c^2 + 3cx = 7a^2c^2.$

10. $5a^2x + 6a^2 = a^2.$

11. $bc - cx = 4bc.$

12. $5(b + x) = 10b.$

13. $6(c - x) + 18c = 0.$

14. $bx - (b + c) = 5b - c.$

15. $x + 2(b - c) = 4c + 2b.$

16. $3ax - ab = 2ax - ac.$

17. $4cy - 3ac = 5ac + 2cy.$

18. $3cy + 2bc = 6bc + 2cy - 3ac.$

19. $4by - 7a^2b = 6ab^2 + 3by.$

20. $ax - 4a = a^2 + 4 - 2x.$

Solution. $ax - 4a = a^2 + 4 - 2x.$

Transposing, $ax + 2x = a^2 + 4a + 4.$

Writing the coefficients of x as a binomial,

$$(a + 2)x = a^2 + 4a + 4.$$

Dividing both members by the coefficient of x,

$$x = \frac{a^2 + 4a + 4}{a + 2}.$$

Performing the division, $x = a + 2$.

Check. Substituting $a + 2$ for x in the original equation,

$$a(a + 2) - 4a = a^2 + 4 - 2(a + 2).$$

Expanding, $a^2 + 2a - 4a = a^2 + 4 - 2a - 4$,

or $a^2 - 2a = a^2 - 2a$.

21. $ax + bx = ac + bc$.

22. $5ax + 4cx = 5ab + 4cb$.

23. $a^2x + 1 - a^4 - x = 0$.

24. $ax + 2ab = 2a^2 + bx$.

25. $ax - a^3 - 4 = 3a - x$.

26. $ax - ac + bc = 2ac - 5bc + 2bx$.

27. $4b^2c^2 + (a + bx)c = (a - bx)c$.

HINT. Perform the indicated multiplications first.

28. $(x + a)(x + b) = x^2 + 2a^2 + 3ab$.

29. $15(x - a) - 6(x + a) = 3(5a - 3x)$.

51. Uniform motion. If a man walks for 8 hours at the rate of 3 miles per hour, he will walk in all 8 × 3, or 24, miles. If a train runs for 12 hours at the rate of 45 miles per hour, the total distance traversed is 12 × 45, or 540, miles. These examples illustrate **uniform motion.** In all problems of uniform motion, the elements involved are

(*a*) Time, measured in seconds, minutes, hours, etc.

(*b*) Rate of motion (velocity), or the distance traveled in a unit of time (one second, one hour, one day, as the case may be).

(*c*) Distance (total), measured in feet, inches, miles, meters, etc.

If a body moves uniformly for a time t, at a rate r, and covers in all a distance d, then the numbers represented by d, r, and t are connected by the equation

$$d = r \times t. \qquad (1)$$

This relation or formula gives an insight into the power of the algebraic method. By means of arithmetical numbers alone we can express the relation which holds between the time, the rate, and the distance only for a particular case. By means of the literal equation (1) we express the relation which is true not merely for one case, but for countless cases. In fact, it holds whenever we are dealing with uniform motion, whatever numerical values d, r, and t may have.

The use of letters enables us in this way to express a law in general terms, and hence to include in one short expression a more perfect idea of the relation implied by uniform motion than could be given by many equations involving only arithmetic numbers.

ORAL EXERCISES

1. Sound travels in air about 1100 feet a second. A soldier observes the shower of earth thrown up by an exploding shell, and 10 seconds later hears the sound of the explosion. How far was he from the place where the shell struck?

2. A man observing a woodman fell a tree hears the sound of the ax three fifths of a second after the blow. How far apart are the two men?

3. How far does an automobile travel if it runs

(*a*) 18 miles per hour for 5 hours?
(*b*) 16 miles per hour for h hours?
(*c*) m miles per hour for h hours?
(*d*) 20 miles per hour for $t+4$ hours?
(*e*) $2x-4$ miles per hour for t hours?

4. What is the rate of an automobile if it runs uniformly

(*a*) 200 miles in 8 hours?
(*b*) m miles in 8 hours?
(*c*) m miles in h hours?
(*d*) 200 miles in $x+3$ hours?

5. An automobile travels a distance d at a rate of 20 miles an hour. Another car runs 50 miles farther at 25 miles an hour. Represent the number of hours required by each car for its trip. If the first car required the same time as the other, what equation involving d can be formed?

6. In a naval engagement the opposing battle cruisers were fighting at a distance of ten miles. If the average velocity of the shells was 2640 feet per second, how many seconds did they remain in the air?

7. An automobile runs 20 miles per hour for t hours, and a second one runs 14 miles per hour for $t + 2$ hours. Represent the distance each travels. Form an equation which states that the distances traveled by each are the same.

8. James travels t miles per hour for 8 hours, and John travels $t - 3$ hours at the rate of 12 miles per hour. Represent the distance each travels. What equation in t may be formed if both travel the same distance?

9. James travels from A to B in t hours at 12 miles an hour. John leaves A just 2 hours after James, and traveling 16 miles per hour reaches B at the same time as James. Represent the distance James travels, the distance John travels, and state the equation in t which may be formed.

10. Two automobiles leave two towns 225 miles apart at the same time and travel toward each other. One travels m miles per hour, while the other travels 5 miles per hour less. They meet in 5 hours. Represent the time, the rate, and the distance for each. What will represent the sum of the distances traveled? What equation in m may be formed?

11. Two cars run 200 miles, one in t hours, the other in 1 hour less time. Represent the rate of each car in miles per hour. If the rate of one is 4 miles per hour more than the other, what equation in t can be formed? If the rate of one is twice the rate of the other?

EXAMPLES IN UNIFORM MOTION

1. A pedestrian traveling 4 miles per hour is overtaken 14 hours after leaving a certain point by a horseman who left the same starting point 8 hours after the pedestrian. Find the rate of the horseman.

Solution. This is a problem in uniform motion, involving the distance, the rate, and the time of a pedestrian and of a horseman respectively. By a careful reading of the problem one discovers that the time for each was a different number of hours, that each went at a different rate, but that each traveled the same distance. Hence the equation will be formed by expressing *d* in terms of *r* and *t* for both the pedestrian and the horseman and then equating the two expressions for *d*.

By the conditions:

	t, or time in hours	r, or rate in miles per hour	Distance in miles, $d = r \times t$
Pedestrian	14	4	$56 = 4 \times 14$
Horseman	6	x	$6 \cdot x$

Hence $6x = 56,$

and $x = 9\frac{1}{3}.$

Check. $4 \cdot 14 = 56;\ 9\frac{1}{3} \cdot 6 = 56.$

Biographical Note. *Sir Isaac Newton.* Sir Isaac Newton (1642–1727) was probably the keenest mathematical thinker who ever lived. He was the son of a farmer of slender means, and as a boy was rather lazy. It is said, however, that his complete victory over a larger boy in a fight at school led him to feel that perhaps he could be equally successful in his studies if he really tried. His ambition and interest being once roused, he never ceased to apply himself during the rest of his long life.

His most important scientific achievement was the discovery and verification of the laws of motion. In his great work called the "Principia" he showed by mathematical reasoning that all bodies, great and small,—the planet revolving around the sun, as well as the apple falling from the tree,—follow the same laws. His greatest discovery in pure mathematics was that of a method called the calculus, which is the basis of most of the advances in mathematics and in theoretical physics made since his time.

SIR ISAAC NEWTON

But important as was Newton's mathematical work, his most significant contribution to mankind was an idea, — the idea that the world in which we live is not independent of the rest of the universe, but that every smallest particle of matter is connected with the most remote planet and star; that we cannot think of the earth as the center of all things, but that we merely occupy our place in a system governed by universal law.

2. Two men, A and B, start from the same place at the same time and travel in opposite directions. B goes twice as fast as A. In 9 hours they are 54 miles apart. Find the rate of each.

One can conveniently represent the conditions of this problem in the form of a table, as follows:

	t, or time in hours	r, or rate in miles per hour	Distance in miles, $d = r \times t$
A	9	r	$9r$
B	9	$2r$	$18r$

Since the men are 54 miles apart at the end of the given time, the sum of the distances traveled is 54.

Hence $$18r + 9r = 54,$$

or $$r = 2, \text{ and } 2r = 4.$$

Check. $4 = 2 \cdot 2;\ 9 \times 2 + 9 \times 4 = 54.$

It should be particularly noted in choosing letters for the unknowns that it is not enough to say, for instance, let x equal the distance, or let t equal the time. This means nothing unless the unit of distance and the unit of time are also stated. The unknown distance is either a number of feet, or miles, or some other unit of length, and the unknown time is a number of seconds, or hours, or some other specific unit of time. A similar remark is pertinent each time a letter is taken to represent the measure of any quantity.

The student should make a table for each of the following problems, similar to those given in the examples.

PROBLEMS

In Problems 1–7, A and B start from the same place at the same time and travel in opposite directions.

1. A goes 6 miles per hour and B goes 9 miles per hour. In how many hours will they be 90 miles apart?

2. A travels three times as fast as B. In 6 hours they are 120 miles apart. Find the rate of each.

3. A travels 4 miles more per hour than B. After 8 hours they are 144 miles apart. Find the rate of each.

4. A goes 3 miles less per hour than B. After 9 hours the distance between them is 189 miles. Find the rate of each.

5. B goes 4 miles less per hour than A and travels two thirds as fast as A. Find the rate of each. After how many hours will the distance between them be 180 miles?

6. A travels 2 hours and stops. B travels 5 hours at a rate double A's rate. Then they are 144 miles apart. Find their rates and the distance each has traveled.

7. Both A and B travel the same distance, B in 9 hours, A in 6. B's rate is 4 miles per hour less than A's. Find the rate of each, and the distance each traveled.

In Problems 8–13, A and B start at the same time from two points 192 miles apart and travel toward each other until they meet. Find the rate of each:

8. If they travel at the same rate and meet in 8 hours.

9. If A travels 2 miles less per hour than B and they meet in 12 hours.

10. If B travels three times as fast as A and they meet in 12 hours.

11. If they meet in 9 hours and B travels 42 miles more than A.

12. If they meet in 6 hours and B goes 4 miles more per hour than A.

13. If they meet in 12 hours and A travels 6 miles more per hour than B.

In Problems 14–16, A and B start at the same time from two points 144 miles apart and travel toward each other until they meet. Find the number of hours from the start until the time of meeting:

14. If B goes 4 miles more per hour than A and travels twice as far as A.

15. If A travels 6 miles per hour and B travels 9 miles per hour, but B is delayed 4 hours on the way.

16. If A is delayed 3 hours and B is delayed 5 hours, and their rates are 16 miles and 8 miles per hour respectively.

17. The distance from Kansas City to St. Louis is 285 miles. A passenger train running 45 miles per hour leaves Kansas City for St. Louis at the same time a freight train running 12 miles per hour leaves St. Louis for Kansas City. In how many hours will they meet?

18. A starts from a certain place and travels 4 miles per hour. Six hours later B starts from the same place and travels in the same direction at the rate of 6 miles per hour. How many hours does B travel before overtaking A?

19. Two bicyclists 108 miles apart start at the same time and travel toward each other. One travels 10 miles per hour, the other 12 miles per hour. The latter is delayed 2 hours on the way. In how many hours will they meet, and how far has each traveled?

20. A passenger train starts 2 hours later than a freight train, from the same station but in an opposite direction. The rate of the passenger train is 42 miles per hour and the rate of the freight train is 24 miles per hour. In how many hours after the passenger train starts will the two trains be 246 miles apart?

21. A messenger going at the rate of 8 miles per hour has journeyed 2 hours when it is found necessary to change the message. At what rate must a second messenger then travel to overtake the first in 8 hours?

22. A man having 4 hours at his disposal wished to ride as far out of town as possible on a trolley car whose rate is 10 miles per hour, and to return on foot at the rate of 3 miles per hour. On the way back he can take a short cut and save one mile. How long a time may he ride on the car?

The velocity of a bullet continually decreases from the instant it leaves the gun. This is due to the resistance of the air. In the following problems consider the velocity of sound to be 1100 feet per second:

23. Two and one-half seconds after a marksman fires his rifle he hears the bullet strike the target, which is 550 yards distant. Find the average velocity of the bullet.

24. One and three-quarters seconds after a marksman fires his revolver he hears the bullet strike the target 50 rods distant. Find the average velocity of the bullet.

CHAPTER XII

IMPORTANT SPECIAL PRODUCTS

52. The square of a binomial. The multiplication

$$\begin{array}{l} a+b \\ \underline{a+b} \\ a^2+\ \ ab \\ \underline{\quad +\ \ ab+b^2} \\ a^2+2\,ab+b^2 \end{array}$$

gives the formula

$$\boldsymbol{(a+b)^2=a^2+2\,ab+b^2.}$$

This may be expressed in words as follows:

I. The square of the sum of two terms is the square of the first term plus twice the product of the two terms plus the square of the second term.

Similarly, $\boldsymbol{(a-b)^2=a^2-2\,ab+b^2,}$

which may be expressed in words as follows:

II. The square of the difference of two terms is the square of the first term minus twice the product of the two terms plus the square of the second term.

Study the application of I and II in the following:

EXAMPLES

1. $(x+1)^2=x^2+2\,x+1.$ **3.** $(3+a)^2=9+6\,a+a^2.$

2. $(y-2)^2=y^2-4\,y+4.$ **4.** $(5-m)^2=25-10\,m+m^2.$

5. $(3\,a+b)^2=9\,a^2+6\,ab+b^2.$

ORAL EXERCISES

Expand the following by I or II, page 105:

1. $(a+1)^2$.
2. $(b+2)^2$.
3. $(x-5)^2$.
4. $(y+10)^2$.
5. $(z-12)^2$.
6. $(2+x)^2$.
7. $(6+x)^2$.
8. $(9-d)^2$.
9. $(11-y)^2$.
10. $(x+a)^2$.
11. $(y+b)^2$.
12. $(m-n)^2$.
13. $(a-c)^2$.
14. $(a+2b)^2$.
15. $(3a-x)^2$.
16. $(5x-y)^2$.
17. $(7l+m)^2$.
18. $(12m-n)^2$.

State two equal binomials whose product is:

19. m^2+4m+4.
20. $x^2+10x+25$.
21. a^2-6a+9.
22. $d^2-12d+36$.
23. $h^2+14h+49$.
24. $64-16m+m^2$.

Find the value of the following:

25. $(5+2)^2$.

Solution. $(5+2)^2=5^2+2\cdot5\cdot2+2^2=25+20+4=49$.

26. $(10+1)^2$.
27. $(9+2)^2$.

28. $(10-3)^2$.

Solution. $(10-3)^2=10^2-2\cdot10\cdot3+3^2=100-60+9=49$.

29. $(12-2)^2$.
30. $(14-4)^2$.
31. $(21)^2$.

HINT. $(21)^2=(20+1)^2$ etc.

32. $(32)^2$.
33. $(53)^2$.
34. $(103)^2$.
35. $(109)^2$.
36. $(201)^2$.
37. $(504)^2$.
38. $(29)^2$.

HINT. $(29)^2=(30-1)^2$ etc.

39. $(48)^2$.
40. $(98)^2$.
41. $(998)^2$.

EXERCISES

Expand the following:

1. $(2c-3d)^2$.
2. $(5x-2y)^2$.
3. $(3a-2b^2)^2$.
4. $(4x+5y^3)^2$.
5. $(7x^2+2y)^2$.
6. $(5x^2-3y)^2$.
7. $(9a^2+2b^2)^2$.
8. $(11a^2b+2)^2$.
9. $(12ab^2-3c)^2$.
10. $(5a^2b-2a^2)^2$.
11. $(-2a^2c^5+3cd^2)^2$.
12. $(-4xy^2-3xz^2)^2$.

Make any changes which are necessary in the following so that each numerator will be the square of its denominator:

13. $\frac{x^2 + 2xy + y^2}{x + y}$.

14. $\frac{m^2 - mn + n^2}{m - n}$.

15. $\frac{a^2 + 10a + 25}{a + 5}$.

16. $\frac{x^2 + 12x + 36}{x - 6}$.

17. $\frac{4 + 4x + x^2}{2 + x}$.

18. $\frac{25 - 10x - x^2}{5 - x}$.

19. $\frac{m^2 + 16m - 64}{8 + m}$.

20. $\frac{a^4 + 10a^2 - 25}{a^2 + 5}$.

21. $\frac{9x^2 - 6x + 1}{3x + 1}$.

22. $\frac{4a^2 + 12ab + 9b^2}{2a - 3b}$.

23. $\frac{4x^4 - 20x^2y + 25y^2}{2x^2 - 5y}$.

24. $\frac{4x^4 - 20x^2y + 25y^2}{5y - 2x^2}$.

53. The product of the sum and the difference of two terms. The multiplication

$$\begin{array}{lll} a & +b & \\ a & -b & \\ \hline a^2 & +ab & \\ & -ab & -b^2 \\ \hline a^2 & & -b^2 \end{array}$$

gives the formula $\boldsymbol{(a + b)(a - b) = a^2 - b^2}$.

This may be expressed in words as follows:

III. The product of the sum and the difference of two terms equals the difference of their squares taken in the same order as the difference of the terms.

The pupil should study the application of III in the following:

EXAMPLES

1. $(x + 1)(x - 1) = x^2 - 1$.

2. $(5 - x)(5 + x) = 25 - x^2$.

3. $(a + x)(a - x) = a^2 - x^2$.

4. $(2c - y)(2c + y) = 4c^2 - y^2$.

5. $(c^2 - 3)(c^2 + 3) = c^4 - 9$.

ORAL EXERCISES

Perform the following indicated operations:

1. $(a+2)(a-2)$.

2. $(m+7)(m-7)$.

3. $(h+9)(h-9)$.

4. $(6+t)(6-t)$.

5. $(1-x)(1+x)$.

6. $(c+d)(c-d)$.

7. $(x+3y)(x-3y)$.

8. $(5x+2y)(5x-2y)$.

9. $(1+4x)(1-4x)$.

10. $(9a-1)(9a+1)$.

11. $(2+3r)(2-3r)$.

12. $(x^2-4)(x^2+4)$.

13. $(7x^2+2y)(7x^2-2y)$.

14. $(4x^2+3x^3y)(4x^2-3x^3y)$.

15. $(2a+3bc^2)(2a-3bc^2)$.

16. $(a+3a^2b)(a-3a^2b)$.

17. $(7a+2x)(2x-7a)$.

18. $(5x+3y)(3y-5x)$.

19. $(5m^2-4n)(5m^2+4n)$.

20. $(2ab+3c)(2ab-3c)$.

21. $(9cd^2-2)(9cd^2+2)$.

22. $(10mn^2-3p)(10mn^2+3p)$.

23. $(4a^2-9b^2)\div(2a+3b)$.

24. $(9x^2-25y^2)\div(3x-5y)$.

25. $(4a^2b^2-c^2)\div(2ab-c)$.

26. $(49-x^4)\div(7+x^2)$.

27. $(25x^4-4)\div(5x^2+2)$.

28. $(36a^4-49)\div(7+6a^2)$.

State two binomials whose product is:

29. m^2-9.

30. c^2-16.

31. $49-m^2$.

32. m^2-49.

33. $81-p^2$.

34. $100-h^2$.

35. a^2-4x^2.

36. x^2-9y^2.

37. $25x^2-4y^2$.

38. $9a^2-16b^2$.

39. $36c^2-1$.

40. $1-49x^2$.

Find the value of:

41. $(12-2)(12+2)$.

42. $(40+3)(40-3)$.

43. $(30+1)(30-1)$.

44. $(50-2)(50+2)$.

45. 22×18.

HINT. $22\times 18=(20+2)(20-2)$ etc.

46. 13×7.

47. 23×17.

48. 43×37.

49. 64×56.

50. 71×69.

51. 83×77.

52. 96×104.

53. 91×109.

54. The product of two binomials having a common term. The multiplication

$$\begin{array}{l}
x + a \\
\underline{x + b} \\
x^2 + ax \\
 + bx + ab \\
\hline
x^2 + (a + b)x + ab
\end{array}$$

gives the formula

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

This may be expressed in words as follows:

IV. *The product of two binomials having a common term equals the square of the common term, plus the algebraic sum of the unlike terms multiplied by the common term, plus the algebraic product of the unlike terms.*

The pupil should study the application of IV in the following:

EXAMPLES

1. $(x + 1)(x + 2) = x^2 + (1 + 2)x + 2 = x^2 + 3x + 2.$

2. $(c - 2)(c - 3) = c^2 + (-2 - 3)c + 6 = c^2 - 5c + 6.$

3. $(n - 5)(n + 2) = n^2 + (-5 + 2)n - 10 = n^2 - 3n - 10.$

ORAL EXERCISES

Multiply the following mentally:

1. $(x + 2)(x + 1).$
2. $(c + 3)(c + 2).$
3. $(h + 4)(h + 5).$
4. $(a + 5)(a + 3).$
5. $(a + 7)(a + 2).$
6. $(c + 5)(c + 4).$
7. $(x - 2)(x - 3).$
8. $(c - 3)(c - 4).$
9. $(x - 5)(x - 2).$
10. $(x - 7)(x - 8).$
11. $(c - 8)(c + 9).$
12. $(c + 10)(c - 3).$
13. $(c - 7)(c + 5).$
14. $(x + a)(x + c).$
15. $(x + k)(x + l).$
16. $(x + a)(x - c).$

EXERCISES

Write the quotients for the following by inspection and check by multiplication:

1. $\frac{x^2+5x+6}{x+3}$.
2. $\frac{a^2+7a+10}{a+2}$.
3. $\frac{a^2+11a+10}{a+10}$.
4. $\frac{b^2+7b+6}{b+1}$.
5. $\frac{c^2-6c+8}{c-2}$.
6. $\frac{c^2-9c+14}{c-7}$.
7. $\frac{x^2-3x+2}{x-2}$.
8. $\frac{x^2-4x-12}{x+2}$.
9. $\frac{a^2-3a-28}{a-7}$.
10. $\frac{a^2-3ab+2b^2}{a-b}$.
11. $\frac{x^2-xy-6y^2}{x-3y}$.
12. $\frac{m^2-4mn-21n^2}{m+3n}$.
13. $\frac{r^2-7rs-18s^2}{r-9s}$.
14. $\frac{r^2-rs-90s^2}{r-10s}$.

55. The product of two binomials of the form $(ax+b)(cx+d)$. The multiplications

$$\begin{array}{r} 2x+5 \\ 3x+7 \\ \hline 6x^2+15x \qquad \\ +14x+35 \\ \hline 6x^2+29x+35 \end{array} \quad \text{and} \quad \begin{array}{r} ax+b \\ cx+d \\ \hline acx^2+\ bcx \qquad \\ +\ adx+bd \\ \hline acx^2+(bc+ad)x+bd \end{array}$$

show that in the product of any two binomials of the form $(ax+b)(cx+d)$

(*a*) the first term is the product of the first terms of the binomials,

(*b*) the second term is the sum of the cross products, and

(*c*) the third term is the algebraic product of the second terms of the binomials.

EXAMPLES

1. $(\underbrace{3x + \overbrace{2)(4}x + 5})=12x^2 + 8x + 15x + 10$

$= 12x^2 + 23x + 10.$

2. $(\underbrace{5a + \overbrace{1)(3}a - 2})=15a^2 + 3a - 10a - 2$

$= 15a^2 - 7a - 2.$

3. $(\underbrace{4b - \overbrace{3)(b} + 2})= 4b^2 + 5b - 6.$

EXERCISES

Write the products in the following:

1. $(2a + 1)(a + 3)$.
2. $(3a + 2)(a + 5)$.
3. $(x + 4)(2x + 3)$.
4. $(2x + 3)(x + 7)$.
5. $(2c + 3)(c + 8)$.
6. $(4c + 3)(4c + 3)$.
7. $(5m + 2)(5m + 2)$.
8. $(4m - 1)(2m + 3)$.
9. $(5n - 3)(3n - 5)$.
10. $(2n + 7)(3n - 2)$.
11. $(7h + 3)(4h - 5)$.
12. $(5h - 4)(5h - 4)$.
13. $(5k - 6)(5k - 6)$.
14. $(3 + m)(4 + 2m)$.
15. $(7 - n)(7 - 3n)$.
16. $(2a - b)(3a + 2b)$.
17. $(5a + 9c)(2a - 9c)$.
18. $(2x^2 - 7y)(3x^2 - 2y)$
19. $(c^2 - 5d)(5c^2 + 2d)$.
20. $(4c + 9d^2)(2c - 5d^2)$.

ORAL REVIEW EXERCISES

Perform the following indicated operations:

1. $(m + 2r)^2$.
2. $(h - 3k)^2$.
3. $(2a + 13)^2$.
4. $(6s - 5r)^2$.
5. $(a^2 + 3x)^2$.
6. $(2m - 3n^2)^2$.
7. $(a + 3c)(a - 3c)$.
8. $(5a + 1)(5a - 1)$.
9. $(5n^2 - p^2)^2$.
10. $(3rs^2 + 2s)^2$.
11. $(2x + 3y)(2x - 3y)$.
12. $(4h^2k - 5h^3)^2$.

13. $(5x^2 - z)(5x^2 + z)$.

14. $(-5a^2c^3 - 2c)^2$.

15. $(31)^2$.

16. $(49)^2$.

17. $(a + 3)(a + 2)$.

18. $(a - 9)(a - 4)$.

19. $(7m^2n + 3)(7m^2n - 3)$.

20. $(4xy^2 + 5z^2)(4xy^2 - 5z^2)$.

21. $(x + y)(x^2 + y^2)(x - y)$.

Hint. Multiply the product of the first and third binomials by the second binomial.

22. $(a - 2)(a^2 + 4)(a + 2)$.

23. $(m + n)(m^2 + n^2)(m - n)(m^4 + n^4)$.

24. $(x^2 + 1)(x^2 - 1)(x^4 + 1)(x^8 + 1)$.

25. $[(a + b) + c][(a + b) - c]$.

26. $[(x + y) + 2][(x + y) - 2]$.

27. $(x + 2a)(x + a)$.

28. $(2c + 1)(c + 2)$.

29. $(3d^2 + 4)(2d^2 + 5)$.

30. $(x - y)(x + 3y)$.

31. $(c - 3d)(c - d)$.

32. $(h^2 + 2k)(h^2 - 3k)$.

33. $(3a^2 + 4)(3a^2 + 4)$.

34. $(a^2 + 3b)(2a^2 - b)$.

35. $(5x^2 - y)(5x^2 - y)$.

36. $(4c^2 + d^2)(4c^2 - d^2)$.

37. $(4x^2 - 2a^3)(4x^2 - 3a^3)$.

38. $(5x - y^3z^2)(2x + 3y^3z^2)$.

39. $(a^2 + 2am + m^2) \div (a + m)$.

40. $(x^2 - 4y^2) \div (x - 2y)$.

41. $(a^2 - 3a + 2) \div (a - 2)$.

42. $(49 - 14x + x^2) \div (7 - x)$.

43. $(c^2 - 5c + 6) \div (c - 3)$.

44. $(9c^2 - 4) \div (3c + 2)$.

45. $(d^2 - 16d - 17) \div (d + 1)$.

What is the quickest way to multiply:

46. $(x + y)(x + y)(x - y)(x - y)$?

47. $(x^2 + a^2)(x + a)(x^4 + a^4)(x - a)$?

48. $(4x^2 + a^2)(2x + a)(16x^4 + a^4)(2x - a)$?

CHAPTER XIII

FACTORING

56. Definitions. **Factoring** is the process of finding two or more expressions whose product is equal to a given expression.

Many simple exercises in factoring were solved in the preceding chapter in connection with the rules of multiplication there given. In fact the process of factoring is the reverse of multiplication.

The subject of factoring is extensive. In this chapter we shall consider only the more common forms of factorable expressions, using only such factors as have integers as coefficients.

If a polynomial cannot be expressed as the product of expressions other than itself and 1, it is said to be **prime.**

57. Square root of monomials. In factoring it is often necessary to find the square root, the cube root, and other roots of monomials.

The **square root** of a monomial is one of the two equal factors whose product is the monomial.

Since $+2 \cdot +2 = 4$ and $-2 \cdot -2 = 4$, the square root of 4 is ± 2, which means both plus 2 and minus 2.

Similarly, the square root of 9 is ± 3 and the square root of a^2 is $\pm a$.

That is, *Every positive number or algebraic expression has two square roots which have the same absolute value but opposite signs.*

It is customary to speak of the positive
a number as the **principal square root,** and if no sig
the radical, the principal root is understood.

Thus $\sqrt{4} = 2$, not -2; $-\sqrt{4} = -2$, not $+2$.

When both the positive and the negative square roots are considered, the double sign must precede the radical.

Since $x^3 \cdot x^3 = (-x^3)(-x^3) = x^6$, then $\pm\sqrt{x^6} = \pm x^3$.

That is, *The exponent of any letter in the square root of a monomial is one half the exponent of that letter in the monomial.*

Hence for extracting the square root of a monomial where both positive and negative factors are desired we have the

Rule. *Write the square root of the numerical coefficient preceded by the double sign ± and followed by all the letters of the monomial, giving to each letter an exponent equal to one half its exponent in the monomial.*

A rule much like the preceding holds for fourth root, sixth root, and other even roots.

Thus $\pm\sqrt[4]{81\,c^8} = \pm 3\,c^2$, and $\pm\sqrt[6]{a^{18}} = \pm a^3$.

In the chapters on Factoring and Fractions, where square roots arise, only the *principal* square root will be considered.

According to the definition of square root, the two factors of a term, either of which is its square root, *must be equal.* Consequently they must have the same sign.

58. Cube root of monomials. The **cube root** of a monomial is one of the three equal factors whose product is the monomial.

Since $3 \cdot 3 \cdot 3 = 27$, $\sqrt[3]{27} = 3$.

And as $-3 \cdot -3 \cdot -3 = -27$, $\sqrt[3]{-27} = -3$.

That is, *The cube root of a monomial has the same sign as the monomial.*

Since $x^5 \cdot x^5 \cdot x^5 = x^{15}$, $\sqrt[3]{x^{15}} = x^5$.

Similarly, as $a^5x^2 \cdot a^5x^2 \cdot a^5x^2 = a^{15}x^6$, $\sqrt[3]{a^{15}x^6} = a^5x^2$.

That is, *The exponent of any letter in the cube root of a term is one third of the exponent of that letter in the term.*

Hence for extracting the cube root of a monomial we have the

Rule. *Write the cube root of the numerical coefficient preceded by the sign of the monomial and followed by all the letters of the monomial, giving to each letter an exponent equal to one third of its exponent in the monomial.*

A rule much like the preceding holds for fifth root, seventh root, and other odd roots.

Thus $\sqrt[5]{-32} = -2$, $\sqrt[5]{x^{10}} = x^2$, and $\sqrt[7]{128\,x^{21}} = 2\,x^3$.

ORAL EXERCISES

Find the value of the following:

1. $\sqrt{3^2}$.
2. $\sqrt{7^2}$.
3. $\sqrt{6^4}$.
4. $\sqrt{5^2 \cdot 2^2}$.
5. $\sqrt{9 \cdot 5^2}$.
6. $\sqrt{4\,x^2}$.
7. $\sqrt{9\,a^4}$.
8. $\sqrt{16\,x^2}$.
9. $\sqrt{25\,x^8}$.
10. $\sqrt{4\,a^2x^6}$.
11. $\sqrt{36\,a^4c^2}$.
12. $\sqrt{64\,m^6n^{10}}$.
13. $\sqrt{49\,r^2s^{14}}$.
14. $\sqrt{100\,a^{10}b^2}$.
15. $\sqrt{144\,a^8y^{10}}$.
16. $\sqrt{196\,h^6k^{16}}$.
17. $\sqrt{121\,a^{20}b^{10}c^2}$.
18. $\sqrt{225\,c^{12}d^6}$.
19. $\sqrt{289\,a^2b^4c^6}$.
20. $\sqrt{361\,x^2y^{12}z^{20}}$.
21. $\sqrt{225\,m^{16}n^8z^4}$.
22. $\sqrt{2^2 \cdot 3^4 \cdot 5^2}$.
23. $\sqrt{10^4 \cdot 6^2 \cdot x^{10}}$.
24. $\sqrt{5^2 \cdot 2^4 \cdot 9}$.
25. $\sqrt{9 \cdot 25 \cdot 81}$.
26. $\sqrt{m^2(n^2)^2}$.
27. $\sqrt{a^6(b^8)^2}$.
28. $\sqrt{4\,x^2(y^2)^2}$.
29. $\sqrt{2^2 \cdot (7^2)^2}$.
30. $\sqrt{3^2 \cdot (2^2)^4}$.
31. $\sqrt{x^4 \cdot (y^{10})^2}$.
32. $\sqrt{25 \cdot 2^6 \cdot r^4}$.
33. $2\sqrt{x^{10}}$.
34. $5\sqrt{a^6 \cdot 2^{10}}$.
35. $7\sqrt{x^2 \cdot 3^2 \cdot y^6}$.
36. $\sqrt[3]{5^3}$.

37. $\sqrt[3]{(-2)^6}$.
38. $\sqrt[3]{(-8)\cdot 2^3}$.
39. $\sqrt[3]{5^3 \cdot 3^6}$.
40. $\sqrt[3]{8 \cdot 3^3}$.
41. $\sqrt[3]{-27 \cdot 8}$.
42. $\sqrt[3]{27 \cdot x^3}$.
43. $\sqrt[3]{-64 \cdot a^9}$.
44. $\sqrt[3]{125\,x^{12}}$.
45. $\sqrt[3]{343 \cdot a^{33}b^3}$.
46. $\sqrt[3]{(-x)^3 \cdot 5^3}$.
47. $\sqrt[3]{(-3)^6 \cdot 8^2 \cdot c^9}$.
48. $\sqrt[3]{1000 \cdot r^6}$.
49. $\sqrt[3]{-729 \cdot 9^3 \cdot s^{12}}$.
50. $\sqrt[4]{a^4b^8}$.
51. $\sqrt[4]{x^{12}y^{16}}$.
52. $\sqrt[4]{2^4x^{20}}$.
53. $\sqrt[4]{3^4 5^8}$.
54. $\sqrt[4]{16}$.
55. $\sqrt[5]{32\,x^{10}}$.
56. $\sqrt[5]{-243\,a^{15}}$.
57. $\sqrt[5]{4 \cdot 2^3 a^{10} b^{15}}$.

59. Polynomials with a common monomial factor. The type form is

$$ab + ac - ad.$$

Since $$ab + ac - ad = a(b + c - d),$$

we have, for factoring expressions having a common monomial factor, the following

Rule. *Determine by inspection the monomial factor which is the product of all numerical and literal factors common to all terms of the polynomial.*

Divide the polynomial by this monomial factor.

Write the quotient in a parenthesis preceded by the monomial factor.

EXAMPLE

Factor $10\,ab^2 - 15\,b^3$.

Solution. The common monomial factor of both terms is seen to be $5\,b^2$. Dividing the binomial by $5\,b^2$, the quotient is $2\,a - 3\,b$.

Therefore $10\,ab^2 - 15\,b^3 = 5\,b^2(2\,a - 3\,b)$.

ORAL EXERCISES

Factor the following:

1. $2x + 2$.
2. $3x + 6$.
3. $2a - 4$.
4. $3c + 9$.
5. $5c - 10$.
6. $7x - 14$.
7. $10x + 20$.
8. $5m - 15$.
9. $10 - 2n$.
10. $12 - 6a$.
11. $2b + 2c$.
12. $5r - 10s$.
13. $6x - 4y$.
14. $10x + 2y$.
15. $16a - 4b$.
16. $20c + 18cd$.
17. $25m - 20mn$.
18. $27x - 36xy$.
19. $30r - 12rs$.
20. $35t + 4st$.

EXERCISES

Write the prime factors of the following:

1. $3ax^2 + 18a$.
2. $8x^3 + 8x$.
3. $2x^4 - 6x^2$.
4. $a^3b^2 + a^3b^3$.
5. $3r^2s - 27r^2$.
6. $10x^3 - 4x^7$.
7. $18x^2 + 27 + 9x$.
8. $x^3 - x^2 + x^4$.
9. $2c^5 - 18c + 2c^3$.
10. $4x^2 - 8ax + 20x$.
11. $5a^2 + 10a^4 - 25a^3$.
12. $3y^4 + 6y^3c - 3y^2$.
13. $a^2 + a^3 - a^4 + 2a$.
14. $5r^2 + 10r + 15r^3$.
15. $6a^5 - 12a^4 + 6a - 18a^6$.
16. $16m^3 - 32m^2n + 24m^2n^2$.
17. $-a^2b - 2ab^2 + a^2b^2$.
18. Solve for x, $ax = a(m + n)$.
19. Solve for x, $ax = ab + ac$.
20. Solve for y, $ny = nr + ns + nt$.
21. Solve for y, $my = mh - mk - ml$.
22. Solve for z, $az = 2a - ab + a^2$.

60. Polynomials which may be factored by grouping terms and taking out a common binomial factor. The type form is

$$ax + ay + bx + by.$$

Plainly $ax + ay + bx + by = a(x + y) + b(x + y)$.

Dividing both terms of $a(x + y) + b(x + y)$ by $(x + y)$, the quotient is $a + b$.

Therefore $ax + ay + bx + by = (x + y)(a + b)$.

EXAMPLE

Factor $2xy + 3ab + 6ay + bx$.

Solution.

$$2xy + 3ab + 6ay + bx =$$
$$2xy + 6ay + bx + 3ab =$$
$$2y(x + 3a) + b(x + 3a) =$$
$$(x + 3a)(2y + b).$$

The preceding example illustrates the

Rule. *Arrange the terms of the polynomial to be factored in groups of two or more terms each, such that in each group a monomial factor may be written outside a parenthesis, which in each case contains the same expression.*

Rewrite, placing these monomial factors outside parentheses.

Then divide by the expression in parenthesis and write the divisor as one factor and the quotient as the other.

Polynomials which may be factored by grouping terms according to the foregoing rule usually contain either four, six, or eight terms.

It is important to note that one can obtain two apparently different sets of factors for a given expression. Thus

$$(m - r)(n - 2x) = (r - m)(2x - n),$$

for each pair by actual multiplication gives $mn - nr - 2mx + 2rx$.

An inspection of the expression shows that the binomials of the first pair are the negatives respectively of those in the second pair; hence either pair of factors is correct.

The relation that the process of factoring bears to the processes of multiplication and division of monomials and polynomials should be constantly kept in mind. In multiplication we have two factors given and are required to find their product. In division we have the product and one factor given and are required to find the other factor. In factoring, however, the problem is a little more difficult, for we have only the product given, and our experience is supposed to enable us to determine the factors. For this reason a very careful study of various typical forms of products is necessary.

There is no simple operation the performance of which makes us sure that we have found the *prime* factors of a given expression. Only insight and experience enable us to find prime factors with certainty.

A partial check, however, that may be applied to all the exercises in factoring, consists in actually multiplying together the factors that have been found. The result should be the original expression.

ORAL EXERCISES

Factor the following:

1. $5(x+y)+c(x+y)$.
2. $a(2b+c)+c(2b+c)$.
3. $m(a+3x)-n(a+3x)$.
4. $2x(x-2y)-y(x-2y)$.
5. $3r(s+7t)-(s+7t)$.
6. $2m(m+n)+3n(n+m)$.
7. $3x(2x-y)-5y(2x-y)$.
8. $m(m-5n)-2n(m-5n)$.
9. $2a(a-b)-3b(a-b)$.
10. $5x(x-3)+3(x-3)$.
11. $4r(r-7)-7(r-7)$.
12. $7s(s-t)-2t(s-t)$.
13. $9m(m-2n)+4n(m-2n)$.
14. $7x(3x-4y)-2y(3x-4y)$.
15. $11ab(a-3b)-5c(a-3b)$.

EXERCISES

Factor the following:

1. $2a(a-x)+3(x-a)$.

Hint. This can be written $2a(a-x)-3(a-x)$.

2. $7m(m-2n)-2(2n-m)$.
3. $3h(5h-k)-k(k-5h)-hk(5h-k)$.
4. $7x(2x-5y)-2y(2x-5y)+(5y-2x)$.
5. $(3a-2x)+9a(-2x+3a)+x(2x-3a)$.
6. $ah+bh+ak+bk$.

7. $mr + nr + ms + ns.$

8. $2ac + bc + 2ad + bd.$

9. $3ac + cx + 9ay + 3xy.$

10. $6ax + 15bx + 4ay + 10by.$

11. $2a^3 + a^2x + 2ax + x^2.$

12. $6h^3 + 4h^2k + 3hk^2 + 2k^3.$

13. $mh - nh + mk - nk.$

14. $2ax - 6ay + bx - 3by.$

15. $ar - cr + as - cs.$

16. $mr - nr + ms - ns.$

17. $2hx - 2kx + hy - ky.$

18. $3x^3 - 3x^2y + xy - y^2.$

19. $2a^3 - 6a^2b^2 + ab^3 - 3b^5.$

20. $5r^3 - 2rs^2 + 10r^2s - 4s^3.$

21. $20x^3 - 5x^2y^2 + 4xy - y^3.$

22. $6ac - 3c + 2ad - d.$

23. $8ax + 15by - 12ay - 10bx.$

Solution.

$$\begin{aligned}&8ax + 15by - 12ay - 10bx =\\&8ax - 12ay - 10bx + 15by =\\&4a(2x - 3y) - 5b(2x - 3y) =\\&(2x - 3y)(4a - 5b).\end{aligned}$$

24. $ac - bc - ad + bd.$

25. $ac - ad - bc + bd.$

26. $2am - 6bm - 3an + 9bn.$

27. $3hx - 15kx - hy + 5ky.$

28. $x^3 - 3xy - 2x^2y^3 + 6y^4.$

29. $2a^3 - 6a^2b^2 - ab^2 + 3b^4.$

30. $10a^4 - 25a^3b^3 + 5b^4 - 2ab.$

31. $14x^4 - 35ax^2 + 10a - 4x^2.$

32. $30x^5 - 10x^2 + 1 - 3x^3.$

33. $ab^4 - 2b^3 - 5ab + 10.$

34. Solve for x, $x(a + b) = r(a + b) + s(a + b).$

35. Solve for x, $mx + nx = mr + nr + ms + ns.$

HINT. In the preceding equation, and in similar ones, the value of the unknown should be obtained by the use of factoring. Mental division should be employed, not ordinary long division.

36. Solve for y, $ay + by = ac + bc - ab - b^2.$

37. Solve for z, $kz - lz = hk - hl - k^2 + lk.$

38. Solve for y, $cy + ad - ae = dc - ce + ay.$

39. Solve for m, $2k + hl - mk = kl + 2h - mh.$

61. Trinomials which are perfect squares. Here the type form is

$$a^2 \pm 2ab + b^2.$$

This, by page 105, gives us the two expressions:

$$a^2 + 2ab + b^2 = (a+b)^2,$$
$$a^2 - 2ab + b^2 = (a-b)^2.$$

If an algebraic expression is the product of two equal factors, it is said to be a **perfect square.**

A trinomial, arranged according to the descending powers of one letter, *is a perfect square if the first and third terms are positive and if the absolute value of the middle term is twice the product of the absolute values of the square roots of the other two terms.*

Thus in the type form above, the middle term $2ab = 2 \cdot \sqrt{a^2} \cdot \sqrt{b^2}$.

Similarly, the trinomial $4x^2 - 20xy^2 + 25y^4$ is a perfect square, since the middle term $20xy^2 = 2 \cdot \sqrt{4x^2} \cdot \sqrt{25y^4} = 2 \cdot 2x \cdot 5y^2$.

ORAL EXERCISES

Form perfect trinomial squares by supplying the missing terms in the following:

1. $a^2 + (?) + x^2$.
2. $c^2 + (?) + 25$.
3. $s^2 + (?) + 16t^2$.
4. $h^2 + (?) + 1$.
5. $1 + (?) + 4h^2$.
6. $9x^2 - (?) + 4$.
7. $c^2 + 2cd + (?)$.
8. $a^2 - 2ax + (?)$.
9. $y^2 - 6y + (?)$.
10. $r^2 + 2r + (?)$.
11. $x^2 - 4x + (?)$.
12. $x^2 + 10x + (?)$.
13. $c^2 - 8c + (?)$.
14. $n^2 - 12n + (?)$.
15. $n^2 - 20n + (?)$.
16. $x^2 - 18x + (?)$.
17. $x^2 - 26x + (?)$.
18. $4r^2 - 4r + (?)$.
19. $9x^2 + 6x + (?)$.
20. $9a^2 - 6a + (?)$.

21. $16x^2 - 24x + (?)$.

22. $16x^2 - 40x + (?)$.

23. $25x^2 - 60x + (?)$.

24. $(?) + 10r + 25$.

25. $(?) - 18x + 81$.

26. $(?) - 8x + x^2$.

For obtaining *one* of the two equal factors of a perfect trinomial square we have the

Rule. *Arrange the terms of the trinomial according to the descending powers of some letter in it.*

Extract the square root of the first and third terms and connect the results by the sign of the middle term.

Before applying the foregoing rule one should never forget to observe whether the expression to be factored is a perfect trinomial square or not.

ORAL EXERCISES

Factor the following:

1. $x^2 + 4x + 4$.

2. $b^2 + 6b + 9$.

3. $c^2 - 8c + 16$.

4. $a^2 + 10a + 25$.

5. $d^2 - 10d + 25$.

6. $h^2 - 12h + 36$.

7. $49 - 14x + x^2$.

8. $64 + 16m + m^2$.

9. $81 - 18m + m^2$.

10. $n^2 - 20n + 100$.

11. $k^2 - 22k + 121$.

12. $144 - 24y + y^2$.

13. $c^2 + 2c + 1$.

14. $1 - 2x + x^2$.

15. $4x^2 + 4x + 1$.

16. $9c^2 + 6c + 1$.

17. $9c^2 + 12c + 4$.

18. $16x^2 + 8x + 1$.

19. $16x^2 - 24x + 9$.

20. $25x^2 - 20x + 4$.

21. $9 + 42x + 49x^2$.

22. $16 - 24x + 9x^2$.

23. $25 - 40x + 16x^2$.

24. $49 + 70x + 25x^2$.

EXERCISES

Write the factors for the following:

1. $9r^2 + 6rs + s^2$.
2. $4a^2 + 4ax + x^2$.
3. $9x^2 - 6xy + y^2$.
4. $16x^2 - 8xm + m^2$.
5. $m^2 + 10mn + 25n^2$.
6. $25y^2 - 10yx + x^2$.
7. $81a^2 + 18ab + b^2$.
8. $m^2 - 26mn + 169n^2$.
9. $9h^2 - 60hk + 100k^2$.
10. $121a^2 - 44ab + 4b^2$.
11. $81x^2 + 126xy + 49y^2$.
12. $36x^2 + 25y^2 - 60xy$.
13. $169a^2 + 9b^2 - 78ab$.
14. $49d^2 + 210cd + 225c^2$.
15. $196a^2 - 140ab + 25b^2$.
16. $9a^2b^2 - 12ab + 4$.
17. $16c^2x^2 + 56cx + 49$.
18. $9m^2n^2 - 24mnp + 16p^2$.

It is only in the beginning of factoring that polynomials are classified for the student. In the practical work of handling fractions and solving equations he must determine for himself the type of the polynomial to be factored. It is therefore very important that he fix in mind the various types and the manner of factoring each. Moreover, he should remember that the polynomials which arise in practice often have three or more factors. Miscellaneous review exercises afford excellent practice in recognizing types and in determining *all* the prime factors.

At this point the suggestions given on page 135 will prove helpful, though only the first three of the types there given have as yet been considered.

REVIEW EXERCISES

Separate into prime factors:

1. $x^3 + 6x^2 + 9x$.
2. $2x^2 + 4x + 2$.
3. $x^3 - 10x^2 + 25x$.
4. $a^3 + 2a^2b + ab^2$.
5. $a^3 + 2a^2x + ax^2$.
6. $2c^3 - 20c^2 + 50c$.
7. $100x - 80x^2 + 16x^3$.
8. $98c + 28c^2 + 2c^3$.

9. $80r - 40r^2 + 5r^3$.

10. $245x^5 - 140x^4 + 20x^3$.

11. $45m^2 - 60mn + 20n^2$.

12. $2ax + 4ax^2 + 2ax^3$.

13. $126c^2d^2 + 147c^3d + 27cd^3$.

14. $2ac + 4cx + 2ay + 4xy$.

15. $2ax - 6bx + 2ay - 6by$.

16. $3mr - 6nr + 15ms - 30ns$.

17. $12ab - 6bc - 6ax + 3cx$.

18. Solve for x, $ax + cx = m(a + c) - n(a + c)$.

19. Solve for x, $(a + b)x = a^2 + 2ab + b^2$.

20. Solve for m, $mh - mk = h^2 - 2hk + k^2$.

21. Solve for n, $nr - 2ns = r^2 - 4rs + 4s^2$.

22. Solve for y, $ky - 4kl - l^2 = 4k^2 - ky - ly$.

23. Solve for y, $acy - ad^2 - ac^2 = ady - 2acd$.

24. Solve for z, $2ade - ae^2 = ad^2 - adz + aez$.

62. A binomial the difference of two squares. The type form is

$$a^2 - b^2.$$

By page 107, $a^2 - b^2 = (a + b)(a - b)$.

Hence we have the

Rule. *Regarding each term of the binomial as positive, extract its square root.*

Add the two square roots for one factor, and subtract the second from the first for the other.

EXAMPLES

1. Factor $c^4 - d^4$.

Solution. $c^4 - d^4 = (c^2 + d^2)(c^2 - d^2)$, by application of the rule;
$= (c^2 + d^2)(c + d)(c - d)$, by application of the rule to $c^2 - d^2$.

2. Factor $81a^8 - 16y^{12}$.

Solution. $81a^8 - 16y^{12} = (9a^4 + 4y^6)(9a^4 - 4y^6)$
$= (9a^4 + 4y^6)(3a^2 + 2y^3)(3a^2 - 2y^3)$.

ORAL EXERCISES

Factor:

1. $c^2 - d^2$.
2. $a^2 - 4$.
3. $h^2 - 1$.
4. $4m^2 - 1$.
5. $9 - r^2$.
6. $1 - k^2$.
7. $1 - 4k^2$.
8. $4x^2 - 25$.
9. $9x^2 - 4y^2$.
10. $16x^4 - y^2$.
11. $9c^2 - 64d^2$.
12. $100a^2 - 121b^2$.
13. $121x^2 - 49y^2$.
14. $144c^2 - 121d^2$.
15. $169a^2 - 100b^2$.
16. $25x^4 - 36y^4$.
17. $36x^4 - 49y^8$.
18. $16x^4 - 121a^2$.
19. $64x^4 - 81z^2$.
20. $100x^2 - 121y^4$.
21. $144x^6 - 169y^8$.

Solve for x,

22. $x(a+3) = a^2 - 9$.
23. $x(c-5) = c^2 - 25$.
24. $x(k+7) = k^2 - 49$.
25. $x(2+c) = 4 - c^2$.

EXERCISES

Factor the following:

1. $a^4 - b^4$.
2. $x^4 - 1$.
3. $a^4 - 16$.
4. $r^4 - 81$.
5. $16a^4 - 1$.
6. $81 - c^4$.
7. $625x^4 - 1$.
8. $625 - a^4$.
9. $625x^4 - 16y^4$.
10. $a^{12} - c^{16}$.
11. $c^4d^4 - 1$.
12. $c^8 - 81$.
13. $c^8 - d^8$.
14. $a^4b^8 - 16$.
15. $x^4y^8 - z^4$.
16. $(x+y)^2 - z^2$.

HINT. $(x+y)^2 - z^2 = [(x+y)+z][(x+y)-z]$ etc.

17. $(x-y)^2 - 25$.
18. $(2x+5)^2 - 4y^2$.
19. $(3a-7)^2 - x^2$.
20. $4(x-9)^2 - 9a^2$.
21. $9(a-b)^2 - 25c^2$.
22. $25x^2(c+d)^2 - 9y^2$.
23. $4a^2 - (2c-d)^2$.

HINT. $4a^2 - (2c-d)^2 = [2a + (2c-d)][2a - (2c-d)]$ etc.

24. $25x^2 - (a-3b)^2$.
25. $49a^2 - (3x+2)^2$.
26. $81a^2 - 4(c^4+2)^2$.
27. $100 - a^2(b+c)^2$.

28. $121\,a^2 - 4\,b^2(c - 3\,d^2)^2$.

29. $y^2z^2 - (y + z)^2$.

30. $c^2e^2 - (c - e)^2$.

31. $c^2d^2 - (c^2 - d)^2$.

32. $9\,a^2b^2 - 4\,(a - b)^2$.

33. $25\,a^2b^2 - 9\,(a + b)^2$.

34. $4\,c^2d^2 - (c - 2\,d)^2$.

35. $16\,c^2d^2 - (a^2 + x)^2$.

36. $25\,c^2d^2 - 3^2(c + d)^2$.

37. $49\,r^2s^2 - 9\,(r^2 - s^2)^2$.

Some polynomials of four or six terms may be arranged as the difference of two squares and factored as in the preceding exercises.

EXERCISES

Factor:

1. $x^2 + 2\,xy + y^2 - z^2$.

Solution.
$$x^2 + 2\,xy + y^2 - z^2 =$$
$$(x + y)^2 - z^2 =$$
$$(x + y + z)(x + y - z).$$

2. $x^2 + 2\,x + 1 - y^2$.

3. $a^2 - 6\,a + 9 - b^2$.

4. $c^2 - 10\,c + 25 - 4\,d^2$.

5. $4\,a^2 + 4\,a + 1 - 9\,b^2$.

6. $16 + 24\,a + 9\,a^2 - 25\,b^2c^2$.

7. $4\,a^2 - 12\,ab + 9\,b^2 - 9\,b^4$.

8. $25\,x^2 + 4\,y^2 - 20\,xy - 16\,z^2$.

9. $9\,x^2 + 25\,y^4 - 81\,z^4 - 30\,xy^2$.

10. $60\,ab - 25\,c^4 + 9\,b^2 + 100\,a^2$.

11. $1 - 14\,ab^2 + 49\,a^2b^4 - b^4$.

12. $4 - 20\,ab^2c^3 + 25\,a^2b^4c^6 - 4\,c^8$.

13. $9\,x^2 - 30\,xy + 25\,y^2 - 16\,z^4$.

14. $16\,x^6 - 8\,x^3y^2 + y^4 - z^4$.

15. $l^2 - m^2 + 2\,mn - n^2$.

Solution.
$$l^2 - m^2 + 2\,mn - n^2 =$$
$$l^2 - (m^2 - 2\,mn + n^2) =$$
$$l^2 - (m - n)^2 =$$
$$[l + (m - n)][l - (m - n)] =$$
$$(l + m - n)(l - m + n).$$

16. $x^2 - 4y^2 + 4yz - z^2$.

17. $x^2 - y^2 + 10yz - 25z^2$.

18. $c^2 - 4a^2 - 12ab - 9b^2$.

19. $4c^2 - a^2 + 10ab - 25b^2$.

20. $9d^2 - a^2 + 6ab - 9b^2$.

21. $16s^2 - 25m^2 + 10mn - n^2$.

REVIEW EXERCISES

Factor:

1. $ax^2 - ay^2$.

2. $a^2c - 4b^2c$.

3. $a^3 + 6a^2b + 9ab^2$.

4. $c^6 + 2c^4 + c^2$.

5. $dx^2 - 25d^3$.

6. $25e^4 - 30e^3 + 9e^2$.

7. $2x^3 - 18xy^2$.

8. $r^7 + 2r^5 + r^3$.

9. $a^3b - 6a^2b^3 + 9ab^5$.

10. $5a^4c^2 - 45c^4$.

11. $a^3x^3 - 2a^2x^2 + ax$.

12. $ax^5 - a^5x$.

13. $2x^5 - 162x$.

14. $x^5y - 16xy^5$.

15. $5a^4 - 5$.

16. $x^5 - xy^4$.

17. $32x^4 - 1250$.

18. $3z(x+y)^2 - 12z^3$.

19. $20x^2 + 20x^3 + 5x^4 + 5x^5$.

20. $8r^4 - 8r^2(3s+t)^2$.

21. $8a^4 - 12a^2bc - 18a^2b^2 - 2a^2c^2$.

22. Solve for x, $x(a+2) = a^2 - 4$.

23. Solve for y, $b^2y + 5y = b^4 - 25$.

24. Solve for z, $z(c^2+25)(c-5) = c^4 - 625$.

25. Solve for m, $mh^2 - mhk = h^3 - hk^2$.

26. Solve for n, $acn + 2c - a^2c = 2cn - 2c$.

63. The quadratic trinomial. The type form is

$$x^2 + bx + c.$$

For many trinomials of this type two binomial factors may be found of the form $(x+r)(x+s)$. The method of factoring to be used is the reverse of the method of multiplying two binomials given on page 109. From a study of the four examples there given it is evident that

(1) *The first term of each binomial factor is the square root of the first term of the trinomial.*

(2) *The second terms of the binomials are those factors of the last term of the trinomial whose algebraic sum equals the coefficient of the middle term of the trinomial.*

EXAMPLES

1. Factor $x^2 + 12x + 32$.

Solution. $x^2 + 12x + 32 = (x + ?)(x + ?)$.

It is necessary to find two numbers whose product is $+32$ and whose sum is $+12$.

Now $32 = 1 \cdot 32 = 2 \cdot 16 = 4 \cdot 8$.

The first two pairs are rejected, for each fails to give the sum $+12$. The third pair of factors of 32, namely 4 and 8, gives the correct sum.

Therefore $x^2 + 12x + 32 = (x + 8)(x + 4)$.

2. Factor $a^2 - 11a + 24$.

Solution. Since 24 is positive, its two factors must have the same sign; since -11 is negative, both factors must be negative. Now $24 = 1 \cdot 24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6$. By inspection of these products, -3 and -8 are found to be the required numbers.

Therefore $a^2 - 11a + 24 = (a - 3)(a - 8)$.

3. Factor $c^2 - c - 42$.

Solution. The product -42 is negative; hence the required factors have *unlike* signs. The sum, -1, being negative, the negative factor of -42 must have the greater absolute value. Now $42 = 1 \cdot 42 = 2 \cdot 21 = 3 \cdot 14 = 6 \cdot 7$. We see that $6 + (-7) = -1$.

Therefore $c^2 - c - 42 = (c + 6)(c - 7)$.

EXERCISES

Factor:

1. $a^2 + 5a + 6$.

2. $c^2 + 7c + 12$.

3. $x^2 + 4x + 4$.

4. $x^2 - 8x + 12$.

5. $m^2 - 13m + 30$.

6. $h^2 - 16h + 15$.

7. $n^2 - 8n + 16$.

8. $k^2 - 17k + 30$.

9. $m^2 - 10m + 25$.

10. $r^2 - 5r - 14$.

11. $s^2 + 2s - 35$.

12. $s^2 - 6s + 9$.

13. $t^2 + t - 42$.

14. $x^2 + 5x - 14$.

15. $10a - 39 + a^2$.

16. $a^4 - 4a^2 - 5$.

17. $a^4 - 14a^2 + 49$.

18. $-x^2 + x + 20$.

Solution. By taking out the factor -1, we may factor as usual.

$$-x^2 + x + 20 = -1(x^2 - x - 20) = -1(x - 5)(x + 4)$$
$$= (5 - x)(x + 4).$$

Note that (-1) is not retained as a factor in the final result.

19. $-x^2 - x + 12$.

20. $-m^2 - 8m + 9$.

21. $-2a + 63 - a^2$.

22. $20 - m - m^2$.

23. $24 + 2n - n^2$.

24. $80 - 2r - r^2$.

25. $x^2 - x - 72$.

26. $a^2 - a - 110$.

27. $a^2 + 12a + 36$.

28. $c^2 - 50 + 5c$.

29. $h^2 - 55 + 6h$.

30. $-4s - 21 + s^2$.

31. $27 + 6x - x^2$.

32. $x^2 + 2rx - 3r^2$.

33. $m^2 + 2mn - 99n^2$.

34. $h^2 - 3hk - 130k^2$.

35. $k^2 - 4kl + 4l^2$.

36. $r^2 + 15rs - 100s^2$.

37. $5hk + k^2 - 36h^2$.

38. $r^2s^2 - 3rst - 40t^2$.

39. $a^2b^2 + 8abc + 7c^2$.

40. $33p^2 - m^2n^2 - 8mnp$.

REVIEW EXERCISES

Factor the following:

1. $x^4 + 7x^3 + 12x^2$.

2. $a^4 - 12a^3 + 36a^2$.

3. $2x^2 + 10x - 48$.

4. $5c^3 + 15c^2 - 140c$.

5. $3m^4 + 66m^3 + 363m^2$.

6. $16a^3 - 4ab^2$.

7. $20c^5 - 5cd^4$.

8. $5x^3 - 20xy^2$.

9. $r^3s + 10r^2s^2 + 21rs^3$.

10. $2h^3k^2 + 2h^2k^3 - 12hk^4$.

11. $2r^5 - 162r$.

12. $45m^3n^2 - 20mn^4$.

13. $2x^3y + 10x^2y^2 + 12xy^3$.

14. $4c^4d + 4c^3d^2 - 24c^2d^3$.

15. $5r^4s - 40r^3s^2 + 60r^2s^3$.

16. $39d^4 - 10cd^3 - c^2d^2$.

17. $18x^5 + 7kx^4 - k^2x^3$.

18. $8abc^5 + a^2b^2c^4 - 65c^6$.

19. $6ah^2 + 3h^2c + 18ahk + 9chk$.

20. $30m^2nr - 10mr + 45m^2n - 15m$.

21. $2a^3 + 4a^2b + 2ab^2 - 2ac^2$.

22. $3c^3 + 6c^2d + 3cd^2 - 12c$.

23. $4x^3 - 4xy^2 - 8xyz - 4xz^2$.

24. Solve for x, $x(a+2) = a^2 + 5a + 6$.

25. Solve for y, $ym - y = m^2 - 4m + 3$.

26. Solve for z, $rz + r = r^2 + 5z - 20$.

27. Solve for r, $ar + 3ac - a^2 = 2cr + 2c^2$.

28. Solve for s, $2as + a^3 + a^4 = 6a^2 + a^2s$.

64. The general quadratic trinomial. The type form is

$$ax^2 + bx + c.$$

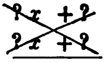
For many trinomials of this type, two binomial factors of the form $(hx + k)(mx + n)$ may be found. The method of factoring such trinomials is illustrated in the following:

EXAMPLES

1. Factor $2x^2 + 5x + 3$.

Solution. $2x^2 + 5x + 3 = (?x + ?)(?x + ?)$.

To get the proper factors we must supply such numbers for the interrogation points in (1) and in (2) as will give (3).



(1)

(2)

$$\begin{array}{r} 2x^2 + ?x \phantom{{}+3} \\ + ?x + 3 \\ \hline 2x^2 + 5x + 3 \end{array} \quad (3)$$

$2x^2$ for the product of the first two terms of the binomials,
$+3$ for the product of the last two terms of the binomials, and
$+5x$ for the sum of the cross products.

Now $\qquad 2x^2 = 2x \cdot x,$

and $\qquad +3 = 1 \cdot 3.$

The factors 1 and 3 may be substituted for the interrogation points in (1) and (2) in either of the following ways:

$$\begin{matrix} 2x+1 \\ x+3 \end{matrix} \text{ (Incorrect)} \qquad \begin{matrix} 2x+3 \\ x+1 \end{matrix} \text{ (Correct)}$$

The first pair is rejected, for it fails to give a product having the required middle term, $+5x$. The second pair gives the correct product.

Therefore $2x^2+5x+3=(2x+3)(x+1)$.

2. Factor $3x^2-x-10$.

Solution. $3x^2=3x\cdot x$

$$-10=1\cdot-10=-1\cdot10=2\cdot-5=-2\cdot5.$$

Test the following pairs of binomials:

$$\begin{matrix} 3x+1 & x+1 & 3x-1 & x-1 & 3x+2 & x+2 & 3x-2 & x-2 \\ x-10 & 3x-10 & x+10 & 3x+10 & x-5 & 3x-5 & x+5 & 3x+5 \end{matrix}$$

Only the last pair gives the desired product.

Therefore $3x^2-x-10=(x-2)(3x+5)$.

After a little practice it will usually be found unnecessary to write down all of the pairs of binomials that do not produce the required product.

If none of the pairs gives the required product, the given trinomial is prime.

EXERCISES

Factor:

1. $3x^2+7x+2$.
2. $2x^2+7x+6$.
3. $3x^2+8x+5$.
4. $2a^2-9a+10$.

Hint. Since the last term is positive and the second negative, only negative factors of 10 need be considered.

5. $5a^2-2a-3$.
6. $6a^2+7a-5$.
7. $10x^2+13x-3$.
8. $6c^2+7c-20$.
9. $10x^2+9x+2$.
10. $6x^2+5x-6$.
11. $2x^2+13x+18$.
12. $3d^2-10d-25$.
13. $5x^2-38x-16$.
14. $10x^2+7x-6$.
15. $4k^2+20k+21$.

16. $9x^2 + 3x - 2$.
17. $16c^2 - 8c - 3$.
18. $10l^2 - 7l - 12$.
19. $12x^2 - 8x - 15$.
20. $14r^2 - 39r + 10$.
21. $21x^2 - 61x - 30$.
22. $25s^2 - 15s + 2$.
23. $36x^2 - 36x + 5$.
24. $36a^2 + 23a - 3$.
25. $49x^2 - 21x + 2$.
26. $2 + x - 15x^2$.
27. $12x^4 + x^2 - 20$.

Hint. $12x^4 + x^2 - 20 = 12(x^2)^2 + x^2 - 20$.

28. $50x^4 + 5x^2 - 3$.
29. $6 + 7x^2 - 5x^4$.
30. $12 + 17x^3 + 6x^6$.
31. $a^2 - 3ab + 2b^2$.
32. $a^2 + 2ab - 8b^2$.
33. $c^2 - cd - 12d^2$.
34. $2x^2 + 5xy + 2y^2$.
35. $2a^2 - 5ab + 2b^2$.
36. $3x^2 - 10xy + 3y^2$.
37. $10x^2 - 27xy + 5y^2$.
38. $12x^2 + 23xy - 2y^2$.
39. $30x^2 - 13xy - y^2$.
40. $30x^2 + 109xy + 30y^2$.

REVIEW EXERCISES

Factor the following:

1. $4x^2 + 10x + 4$.
2. $3x^3 + 18x^2 + 27x$.
3. $20x^4 - 60x^3 + 45x^2$.
4. $50ax^2 - 140ax + 98a$.
5. $27x^3 - 36x^4 + 12x^5$.
6. $8x^4 + 2x^2 - 1$.
7. $60x^3 - 35x^2 - 60x$.
8. $3x^3y - 4x^2y^2 + xy^3$.
9. $12x^5y + 21x^3y^3 - 6xy^5$.
10. $5x^3y + 10x^2y^2 - 75xy^3$.
11. $6x^2r^2 + 12ar^2x + 4rx^2 + 8axr$.
12. $-18ax^2y - 45a^2xy - 6ax^2 - 15a^2x$.
13. $45x^4y^2 + 21x^3y - 6x^2$.
14. Solve for r, $r(a + 1) = 2a^2 + 3a + 1$.
15. Solve for s, $sb + sc = b^2 - bc - 2c^2$.
16. Solve for x, $2hx = 6h^2 + h - 2 + x$.
17. Solve for y, $cy - cd - 2c^2 = dy - 3d^2$.

65. A binomial the sum or the difference of two cubes. The type form is

$$a^3 \pm b^3.$$

a^3+b^3 divided by $(a+b)$ gives the quotient a^2-ab+b^2, and a^3-b^3 divided by $(a-b)$ gives the quotient a^2+ab+b^2.

Therefore $$a^3+b^3=(a+b)(a^2-ab+b^2), \quad (1)$$

and $$a^3-b^3=(a-b)(a^2+ab+b^2). \quad (2)$$

Formulas (1) and (2) above may be expressed in words as follows:

(1) *The factors of the sum of the cubes of two terms are (a) a binomial which is the sum of the terms and (b) a trinomial in which the terms are the square of the first term minus the product of the first and the second term plus the square of the second term.*

(2) *The factors of the difference of the cubes of two terms are (a) a binomial which is the difference of the terms and (b) a trinomial in which the terms are the square of the first term plus the product of the first and the second term plus the square of the second term.*

EXAMPLES

1. Factor a^3+27.

Solution. $$a^3+27=a^3+3^3=(a+3)(a^2-a\cdot 3+3^2)$$
$$=(a+3)(a^2-3a+9).$$

2. Factor $27a^6-64b^3$.

Solution. $$27a^6-64b^3=(3a^2)^3-(4b)^3$$
$$=(3a^2-4b)[(3a^2)^2+(3a^2)(4b)+(4b)^2]$$
$$=(3a^2-4b)(9a^4+12a^2b+16b^2).$$

EXERCISES

Factor the following:

1. $x^3 + y^3$.
2. $m^3 - n^3$.
3. $a^3 + 8$.
4. $b^3 - 64$.
5. $c^3 + 8d^3$.
6. $m^3 - 125n^3$.
7. $8 - x^3$.
8. $64 + y^3$.
9. $27 + 8a^3$.
10. $m^9 - n^3$.
11. $125x^3 - y^3$.
12. $1 - 27m^3n^9$.
13. $8x^3y^6 + 1$.
14. $216 - x^3$.
15. $m^6 + n^6$.
16. $x^6 + y^6$.
17. $x^{12} + y^{12}$.
18. $64x^6 + y^6$.
19. $a^6 + 64$.
20. $m^6 - n^6$.

HINT. This expression may be regarded either as the difference of two cubes, $(m^2)^3 - (n^2)^3$, or as the difference of two squares, $(m^3)^2 - (n^3)^2$. Since the difference of two squares is one of the simplest type forms to factor, one should always use it when possible.

Thus $m^6 - n^6 = (m^3 + n^3)(m^3 - n^3)$ etc.

21. $x^6 - y^6$.
22. $a^6 - 64$.
23. $x^6 - 64y^6$.
24. $1 - a^6$.
25. $64x^6 - 1$.
26. $a^{12} - b^6$.
27. $x^{12} - 1$.
28. $y^{12} - x^6$.

REVIEW EXERCISES

1. $x^4 + xy^3$.
2. $2x^4y - 2xy^4$.
3. $2a^4 - 54a$.
4. $5x^6 - 40x^3$.
5. $x^4 - 2x^3 + 27x - 54$.
6. $x^6 - 7x^3 - 8$.
7. $x^{10} - 64x^4$.
8. $3x^4 + 2x^3 - 24x - 16$.
9. Solve for x, $3x - ax = 27 - a^3$.
10. Solve for x, $bx - 8 = b^3 - 2x$.
11. Solve for x, $c^2x - 27 = c^3 + 3cx - 9x$.
12. Solve for r, $a^3 + a^2r = 8 - 2ar - 4r$.
13. Solve for s, $abs + as + a = ab^3 - ab^2s$.

66. General directions for factoring. Since no general method of factoring can be stated in a few simple rules, the process must be learned by means of such type forms and typical solutions as are given in the preceding pages. When once these have been thoroughly mastered, readiness in factoring expressions which are represented by them becomes a matter of experience. Usually a student finds it comparatively easy to factor a list of exercises classified under a particular type form, yet a list of miscellaneous exercises he finds difficult. This usually indicates inability to determine the type of an expression from its appearance. Until the student, by careful study of the type forms, has acquired the ability to do this, he will make little progress. There are many types that are not included in this book, which the student who continues the study of algebra will meet later.

The following suggestions will prove helpful in solving the types here considered:

I. First look for a common monomial factor, and if there is one (other than 1), separate the expression into its greatest monomial factor and the corresponding polynomial factor.

II. Then by the form of the polynomial factor determine with which of the following types it should be classed and use the methods of factoring applicable to that type.

1. $ax + ay + bx + by$.
2. $a^2 \pm 2ab + b^2$.
3. $a^2 - b^2$.
4. $x^2 + bx + c$.
5. $ax^2 + bx + c$.
6. $a^3 \pm b^3$.

III. Proceed again as in II with each polynomial factor obtained until the original expression has been separated into its prime factors.

MISCELLANEOUS EXERCISES

Factor:

1. $x^3 - 4x$.
2. $2a^3 + 4a^2b + 2ab^2$.
3. $ax^3 + ax^2y + a^2x^2 + a^2xy$.
4. $4x^4 + 400x^8 - 116x^6$.
5. $2cx^3 + 2cx^2 - 12cx$.
6. $5a^3 - 10a^2b - 75ab^2$.
7. $10x^4 - 5x^3 - 30x^2$.
8. $5x^{10} + 40x^7$.
9. $x^3y + 3x^3 + 8y + 24$.
10. $x^5 - 4x^3 - 8x^2 + 32$.
11. $16x^4 - 8x^3 - 2x + 1$.
12. $a^5 - 25a^3 - a^2 + 25$.
13. $-45a^3x^3 - 3a^4x^2 + 18a^5x$.
14. $9x^2 - 30axy^2 + 25a^2y^4$.
15. $x^{12} - 63x^6 - 64$.
16. $x^6 - 64y^{12}$.
17. $6x^4 - 7x^2y - 5y^2$.
18. $3 + 15x - 2ax - 10ax^2$.
19. $x^7 - 27x^4 - x^3 + 27$.
20. $a^3 + 3a^2b - ab^2 - 3b^3$.
21. $20x^4y + 38x^3y^2 - 30x^2y^3$.
22. $8ax^2 - 2a - 20x^2 + 5$.
23. $9a^2 + 4b^2 - 12ab - 4x^2y^2$.
24. $4x^2 - x^4y^4 - 4xy + y^2$.
25. $x^2 + y^2 - 4a^2 - 2xy$.
26. $x^5 - 4x^3y^2 + x^2y^3 - 4y^5$.
27. $x^2 + x^3 - 6x^4$.
28. $1 - x^2 + x^3 - x^5$.
29. $1 - 4x^2 + 8x^3 - 32x^5$.
30. $x^6 - 8x^4 + 16x^2$.
31. $2x^3y + 4x^2y + 2xy - 2xy^3$.
32. $8a^2y - 8ay^2 + 2y^3 - 2y$.
33. $4x^4 + 4x^3 - 4x^6 + x^2$.
34. $a^6 - 13a^4 + 36a^2$.
35. $x^8y^2 - 4x^6y^2 + 3x^4y^6 - 12x^2y^6$.
36. Solve for x, $ax + ad + bd = ac + bc - bx$.
37. Solve for x, $ax + bx + 3 = a + b + 3x$.
38. Solve for x, $10r - 25 - 5x = r^2 - rx$.
39. Solve for x, $4d^2 + 2dx - c^2 = -cx$.
40. Solve for z, $4r^2 + 9 + 3z = 2rz + 12r$.
41. Solve for x, $m + 42 - 7x = m^2 - mx$.
42. Solve for x, $a + 15 + 5x = 2a^2 - 2ax$.
43. Solve for x, $-3c - x = 2c^2 - 2cx - 2$.
44. Solve for x, $6r^2 - 12r + 4x = r^3 - r^2x + 4rx - 8$.
45. Solve for y, $a^3 - c^3 = ay - cy$.

CHAPTER XIV

SOLUTION OF EQUATIONS BY FACTORING

67. Simple equations. A **simple** or **linear equation** in one unknown is one which may be put in such a form that

(*a*) the unknown does not appear in any denominator;
(*b*) only the first power of the unknown is involved.

Thus $3x-4=7$, $2n+5=4n+1$, $ax+b=0$, are simple equations. $(x-1)(x+3)=(x+4)(x-6)$ is also a simple equation, since on multiplying out it becomes $x^2+2x-3=x^2-2x-24$, from which, after transposition, we get $4x+21=0$.

In the preceding chapters only simple equations have been considered.

68. Quadratic equations. A **quadratic equation** in one unknown is one that may be put in such a form that

(*a*) the unknown does not appear in any denominator;
(*b*) the second but no higher power of the unknown is involved.

Thus $x^2+4x-5=0$, $3x^2-4=5x+6$, $ax^2+bx+c=0$, are quadratic equations.

A quadratic equation is often called an equation of the *second degree.*

The term in a quadratic equation which does not involve the unknown is called the constant term.

69. Solution of equations. The methods of factoring given in Chapter XIII enable us to solve many quadratic equations. In the solution of equations by factoring, use is made of the following

Principle. *If the product of two or more factors is zero, one of the factors must be zero.*

Two or more, or even all, of the factors *may* be zero, but the vanishing of one is *sufficient* to make the product zero.

Consider the equation, in factored form,

$$(x-2)(x-4)=0. \tag{1}$$

If this equation is to be solved, all of the numbers which satisfy it must be found. That is, we must find every value of x for which the product on the left of (1) is zero. If the product is to equal zero, the foregoing principle requires that one of the factors be zero. Hence any value of x which satisfies (1) must make either $x-2=0$ or $x-4=0$. Hence x must equal either 2 or 4. On substituting 2 for x in (1) we obtain $(2-2)(2-4)=0$, or $0\cdot(-2)=0$. Hence 2 is a root of (1). On substituting 4 for x in (1) we obtain $(4-2)(4-4)=0$, or $2\cdot 0=0$. Hence 4 is a root of (1). A moment's inspection makes it clear that 2 and 4 are the only roots of the equation.

ORAL EXERCISES

For what value of x is each of the following expressions equal to zero?

1. $x-2$. **3.** $x-8$. **5.** $3x-12$.

2. $x+5$. **4.** $x+10$. **6.** $2x+4$.

7. What is the value of 2×0? of $(-2)\times 0$? of 9×0? of 0×21?

8. Make a general statement which includes all the results of Exercise 7.

What is the value of:

9. $(x-1)(x-2)$ when $x=3$? 1? 0?
10. $(x-4)(x-3)$ when $x=3$? 2? 4?
11. $(x-3)(x-5)$ when $x=-1$? 3? 6?
12. $(2x+1)(3x-1)$ when $x=-1$? $\frac{1}{3}$? 0?
13. $x(x-1)$ when $x=0$? 1? 2?
14. $(x-1)(x-2)(x-3)$ when $x=1$? 2? 3?
15. $3x(x+2)$ when $x=-2$? 0? 2?

In Exercises 16–22, which of the numbers at the right of each equation is a root of that equation?

16. $(x-3)(x-1)=0.$	1, 2, 3.
17. $(x+2)(x-4)=0.$	$-4, -2, 2, 4.$
18. $x(3x-2)=0.$	0, 1, 2.
19. $(2x+1)(x+3)=0.$	$-\frac{1}{2}, -2, -3.$
20. $(x-1)(x-2)(x-3)=0.$	1, 2, 3, 4.
21. $x(x+1)(x-1)=0.$	$-1, 0, 1, 2.$
22. $(x-1)^2(x-9)=0.$	1, 9, 10.

What are the roots of the equations in Exercises 23–28?

23. $(x-1)(x+1)=0.$
24. $(x+2)(x-3)=0.$
25. $(x-2)(x-1)(x+2)=0.$
26. $(2x-3)(x-1)=0.$
27. $(5x+2)(x+2)=0.$
28. $(2x-1)(3x+5)=0.$

29. Is there any one value of x which makes both factors of $(x-3)(x+6)$ equal to zero?

EXAMPLES

1. Solve the quadratic equation $x^2+5x=6$.

Solution. Transposing, $x^2+5x-6=0.$

Factoring, $(x-1)(x+6)=0.$

The value of x which makes the factor $x-1$ equal to zero is a root of the quadratic. Setting $x-1=0$, we obtain $x=1$.

Similarly, the value of x which makes $x + 6$ equal to zero is a root of the quadratic. Setting $x + 6 = 0$, we obtain $x = -6$.

Hence 1 and -6 are the roots of the given quadratic equation.

Check. Substituting 1 for x in $x^2 + 5x = 6$, we have $1 + 5 = 6$.
Substituting -6 for x in $x^2 + 5x = 6$, we have $36 - 30 = 6$.

2. Solve the quadratic equation $x^2 = 4x$.

Solution. Transposing, $x^2 - 4x = 0$.

Factoring, $x(x-4) = 0$.

The factors are x and $x - 4$. The value which makes the first factor zero is $x = 0$. The value of x which makes the second factor zero is $x = 4$. Hence the roots of $x^2 - 4x = 0$ are 0 and 4.

Check. Substituting $x = 0$ in $x^2 = 4x$, $0 = 0$.
Substituting $x = 4$ in $x^2 = 4x$, $16 = 16$.

For solving an equation in one unknown by factoring we have the

Rule. *Transpose the terms so that the right member is zero. Then factor the expression on the left, set each factor which contains the unknown equal to zero, and solve the resulting equations.*

It must be kept in mind that a root of an equation is a number which satisfies the equation.

One should never divide each member of an equation by an expression containing the unknown, for in this manner roots may be lost.

Thus, if in Example 2 we had divided both sides of the equation by x, the resulting equation would have been $x - 4 = 0$, which, to be sure, gives us one root of the given equation. But we have lost the root $x = 0$ which corresponds to the factor by which we divided.

EXERCISES

Find the roots of the following quadratic equations, and check as directed by the teacher:

1. $x^2 - 9 = 0.$
2. $x^2 - 25 = 0.$
3. $x^2 = 16.$
4. $x^2 = 49.$
5. $x^2 - 2x = 0.$
6. $2x^2 + 6x = 0.$
7. $x^2 = 5x.$
8. $3x^2 = 9x.$
9. $x^2 - 4x + 3 = 0.$
10. $x^2 - 6x + 9 = 0.$
11. $x^2 - 7x = -12.$
12. $x^2 + x = 20.$
13. $x^2 + 8 = -6x.$
14. $x^2 = 3x + 10.$
15. $4x^2 - 16x = 0.$
16. $5x^2 + 35x = 0.$
17. $8 - 9x = -x^2.$
18. $12x - 28 = -x^2.$
19. $x^2 - 16x + 64 = 0.$
20. $x^2 - 54 = 15x.$
21. $12 - 25x + 12x^2 = 0.$
22. $3x^2 + x = 4.$
23. $18x^2 = 9x + 20.$
24. $(x + 8)(x + 1) = -12.$
25. $x^2 + 9x - 12 = 3x + 15.$
26. $x + 12 = x^2.$
27. $50x + 24 = 25x^2.$
28. $(x - 11)(x + 3) = 2x + 42.$
29. $x^2 - a^2 = 0.$
30. $x^2 = 4b^2.$
31. $x^2 = 16k^2.$
32. $x^2 - 2bx + b^2 = 0.$
33. $x^2 + 4a^2 = 4ax.$
34. $x^2 - ax = 0.$
35. $x^2 = 7ax.$
36. $2x^2 - 7x = 21 - 6x.$
37. $x^2 - bx - ax = 0.$
38. $x^2 - 5x = 2ax.$
39. $x^2 - ax - bx + ab = 0.$
40. $x^2 + bx = 4b + 4x.$

70. Cubic equations. An equation in x which may be put in the form

$$ax^3 + bx^2 + cx + d = 0,$$

where the coefficients a, b, c, and d represent numbers, is called a **cubic equation**, or an equation of the third degree.

Some equations of higher degree than the second may be solved by the method of factoring.

EXAMPLE

Solve the cubic equation $x^3 - 9x = 9 - x^2$. (1)

Solution. Transposing, $x^3 + x^2 - 9x - 9 = 0$.

Grouping, $x^2(x+1) - 9(x+1) = 0$.

Factoring, $(x+1)(x^2-9) = 0$,

or $(x+1)(x+3)(x-3) = 0$.

Setting each factor equal to zero,

$$x + 1 = 0, \text{ whence } x = -1,$$
$$x + 3 = 0, \text{ whence } x = -3,$$
$$x - 3 = 0, \text{ whence } x = 3.$$

Therefore -1, -3, and 3 are the roots of the equation $x^3 - 9x = 9 - x^2$.

Check. When $x = -1$, (1) becomes $-1 + 9 = 9 - 1$.
When $x = -3$, (1) becomes $-27 + 27 = 9 - 9$.
When $x = 3$, (1) becomes $27 - 27 = 9 - 9$.

EXERCISES

Find the roots of the following equations, and check:

1. $x^3 - 9x = 0$.
2. $3x^3 = 12x$.
3. $x^3 + 4x^2 - 12x = 0$.
4. $x^3 - 14x = 5x^2$.
5. $5x^2 = 2x - 3x^3$.
6. $x^3 - x^2 - 4x + 4 = 0$.
7. $2x^3 - x^2 = 8x - 4$.
8. $3x^3 - 2x^2 - 12x + 8 = 0$.
9. $x^3 - 25 = 25x - x^2$.
10. $2(x^3 - x) = 3(1 - x^2)$.
11. $x^4 - 5x^2 + 4 = 0$.
12. $x^4 - 13x^2 + 36 = 0$.
13. $x^4 = 4x^2$.
14. $x^4 = 26x^2 - 25$.

PROBLEMS

1. The square of a certain number, plus the number itself, is 42. Find the number.

HINT. $n^2 + n = 42$.

2. If from the square of a certain number three times the number be taken, the remainder will be 54. Find the number.

3. If to the square of a certain number the sum of twice the number and 9 be added, the result will be 129. Find the number.

4. Three times the square of a certain number is equal to four times the number. What is the number?

5. A certain number is added to 16, and the same number is also added to 21; the product of the two sums is 546. What is the number?

6. A certain number is subtracted from 15, and the same number is also subtracted from 25; the product of the remainders is 119. Find the number.

7. From 30 a certain number is subtracted, and the same number is added to 18; the product of the results thus obtained is 560. Find the number.

8. If a certain number be added to 15, and the same number be subtracted from 22, the product of the sum and difference thus obtained will be 36 more than 50 times the number. Find the number.

9. If from the square of four times a certain number, five times the number be taken, the result will be 15 times the square of the number. Find the number.

The student has probably observed that a quadratic equation has two roots — one corresponding to each factor which contains an unknown. In Problems 1–9, where the question is asked about a "certain number," both roots of the equation are solutions of the problem, since both are numbers. In problems like the 10th, where the question is asked about some measurement, it frequently happens that one of the roots is a number which cannot measure the particular kind of thing with which the problem deals. Thus one root of the equation to which Problem 10 leads is − 20; and − 20 as the length of a lot is meaningless. Hence we reject it as a solution of the problem, in spite of the fact that it occurs as a root of the equation. Similarly, if a problem dealing with dimensions yields an equation with a negative root, or if a problem asking "how many men" yields an equation with a fractional root, we reject these roots. Although they satisfy the algebraic conditions of the equation to which the problem leads, they fail to comply with the physical conditions of the problem itself, and consequently should not be retained as answers.

10. The depth of a certain lot whose area is 1600 square feet is four times its frontage. Find its dimensions.

11. The area of the floor of a certain room is 54 square yards. The length is 3 yards more than the breadth. What are the dimensions of the floor?

12. The area of a rectangular field is 216 square rods. The field is 6 rods longer than it is wide. Find its dimensions.

13. The sum of the squares of two consecutive integers is 313. Find the numbers.

14. The sum of the squares of two consecutive odd integers is 514. Find the numbers.

15. The sum of the squares of three consecutive odd integers is 251. Find the numbers.

16. An uncovered box 6 inches deep, with square bottom, has 112 square inches of inside surface. Find the other inside dimensions.

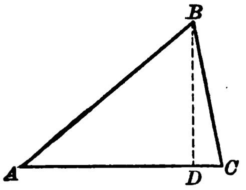
17. Remembering that the faces of a cube are squares, find the edge of a cubical box whose entire outer surface is 216 square inches.

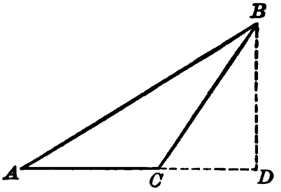
18. A rectangular box is three times as long and twice as wide as it is deep. There are 550 square inches in its entire outer surface. Find its dimensions.

19. A box is 3 inches longer and 1 inch wider than it is deep. There are 62 square inches in its entire outer surface. Find its dimensions.

The *altitude* of a triangle is the perpendicular from any vertex to the side opposite. This side is called the *base*.

In the adjacent figures BD is the altitude and AC is the base of each triangle.





If a is the altitude of a triangle and b its base, the area of the triangle is $\frac{ab}{2}$.

In making use of this and similar formulas the unit in terms of which the lines are measured must be specified.

20. The area of a triangle is 40 square feet; its altitude is 8 feet. Find the base.

21. The altitude of a triangle is twice the base and the area is 36 square feet. Find the base and the altitude.

22. The base of a triangle is 5 times the altitude and the area is 40 square feet. Find the base and the altitude.

23. The area of a triangle is 48 square inches; the base is six times the altitude. Find the altitude and the base.

24. The area of a triangle is 24 square feet; the altitude is 2 feet longer than the base. Find the altitude and the base.

HINT. Let $x =$ the base in feet.
Then $x + 2 =$ the altitude in feet,
and $\frac{x(x+2)}{2} = \frac{x^2 + 2x}{2} =$ the area.
Therefore $\frac{x^2 + 2x}{2} = 24.$

Multiplying each member by 2, this equation becomes

$$x^2 + 2x = 48.$$

25. The altitude of a triangle is 4 feet longer than the base. The area is 6 square feet. Find the base and the altitude.

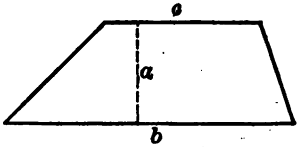
26. One leg of a right triangle is 4 yards longer than the other and the area is 30 square yards. Find the legs.

27. The area of a right triangle is 26 square yards and one leg is 8 feet longer than the other. Find the legs.

28. The area of a triangle is $2\frac{5}{8}$ square feet and the base is 6 inches longer than twice the altitude. Find the base and altitude.

29. The area of a triangle is 12 square yards and the altitude is 6 feet less than twice the base. Find the base and the altitude.

A *trapezoid* is a four-sided figure, two of whose sides are unequal and parallel.



The bases of a trapezoid are the two parallel sides, b and c.

The altitude, a, is the perpendicular distance between the bases.

The area of a trapezoid is given by the formula $\frac{a(b+c)}{2}$.

30. Find the area of a trapezoid whose bases are 10 and 18 and whose altitude is 12.

31. The altitude of a trapezoid is 8 inches, its area is 96 square inches, and one base is 4 inches longer than the other. Find each base.

HINT. Let x = the length of one base in inches.

Then $x + 4$ = the length of the other base in inches,

and $\frac{8(x + x + 4)}{2}$, or $8x + 16$ = area of the trapezoid.

Therefore $8x + 16 = 96$.

32. One base of a trapezoid is 12 feet, the other base is three times the altitude, and the area is 90 square feet. Find the altitude.

33. The altitude of a trapezoid is one half the shorter base and the latter is two thirds of the other base. The area is 250 square feet. Find the bases and the altitude.

34. One base of a trapezoid is 12 feet longer than the altitude, the other base is 6 feet longer than the altitude, and the area is 112 square feet. Find the bases and the altitude.

35. The bases of a trapezoid are respectively 1 foot and 5 feet longer than the altitude, and the area is 30 square yards. Find the bases and the altitude.

36. One base of a trapezoid is 6 feet longer than the other, the altitude is one half the sum of the bases, and the area is 9 square yards. Find the bases and the altitude.

37. The area of a trapezoid is 8 square yards, the altitude equals one base, and the other base exceeds the altitude by 2 feet. Find the bases and the altitude.

38. One base of a trapezoid exceeds the other by 10 feet, the altitude is 2 feet longer than five times the shorter base, and the area is 22 square yards. Find the altitude and the two bases.

CHAPTER XV

FRACTIONS

71. Algebraic fractions. The expression $\frac{a}{b}$, in which a and b represent numbers or polynomials, is an **algebraic** fraction. It is read "a divided by b," or "a over b." A fraction is an indicated quotient in which the dividend is the numerator and the divisor the denominator. The numerator and denominator are often called the *terms* of a fraction.

Certain operations upon fractions, such as multiplying both numerator and denominator by a number (raising to higher terms), and dividing both numerator and denominator by a number (reducing to lower terms), are often necessary before the processes of addition or subtraction of two or more fractions can be performed.

The change of a fraction to lower or to higher terms, and the addition and the subtraction of fractions in both arithmetic and algebra, depend on the

Principle. *The numerator and the denominator of a fraction may be multiplied by the same expression or divided by the same expression without changing the value of the fraction.*

Thus $\frac{3}{4}=\frac{3\cdot 4}{4\cdot 4}=\frac{12}{16}$, and $\frac{18}{30}=\frac{18\div 6}{30\div 6}=\frac{3}{5}$.

Similarly, $\frac{a}{b}=\frac{a\cdot n}{b\cdot n}=\frac{an}{bn}$, and $\frac{a}{b}=\frac{a\div n}{b\div n}=\frac{a/n}{b/n}$.

Since $\frac{4}{4}$, $\frac{6}{6}$, and $\frac{n}{n}$ are each equal to 1, each of the four preceding illustrations is really a multiplication or a division of a fraction by 1. This produces no change in the numerical value of *any* fraction, though it may change its form.

ORAL EXERCISES

Read the result of multiplying both numerator and denominator of each fraction by the factor on its right:

1. $\frac{1}{3}$. 2.

2. $\frac{2}{5}$. 3.

3. $\frac{3}{7}$. 5.

4. $\frac{a}{z}$. 3.

5. $\frac{5}{x}$. 4.

6. $\frac{a}{b}$. x.

7. $\frac{2}{a+x}$. a.

8. $\frac{a-x}{a+x}$. ax.

How are the fractions on the right obtained from those which precede?

9. $\frac{1}{2}, \frac{3}{6}$.

10. $\frac{1}{5}, \frac{3}{15}$.

11. $\frac{2}{5}, \frac{8}{20}$.

12. $\frac{3}{7}, \frac{6}{14}$.

13. $\frac{4}{9}, \frac{8a}{18a}$.

14. $\frac{7}{10x}, \frac{21x}{30x^2}$.

Read the result of dividing both numerator and denominator of each fraction by the number on its right:

15. $\frac{2}{4}$. 2.

16. $\frac{3}{9}$. 3.

17. $\frac{5}{20}$. 5.

18. $\frac{12}{18}$. 6.

19. $\frac{2a}{6}$. 2.

20. $\frac{a}{ab}$. a.

21. $\frac{2x}{x+xy}$. x.

22. $\frac{a+ab}{a-ab}$. a.

How are the fractions on the right obtained from those which precede?

23. $\frac{4}{10}, \frac{2}{5}$.

24. $\frac{12}{18}, \frac{2}{3}$.

25. $\frac{24}{30}, \frac{4}{5}$.

26. $\frac{a^2}{ab}, \frac{a}{b}$.

27. $\frac{xy^2}{x^2y}, \frac{y}{x}$.

28. $\frac{2x^3}{6x^2y}, \frac{x}{3y}$.

72. Reduction of fractions to lowest terms. A fraction is in its **lowest terms** when no factor except 1 is common to both numerator and denominator.

Cancellation is the process of dividing the numerator and the denominator of a fraction by a factor common to both.

EXAMPLES

Reduce to lowest terms:

1. $\frac{54a^3b^5c^2}{144ab^2c^4}$.

Solution
$$\frac{54a^3b^5c^2}{144ab^2c^4}=\frac{\cancel{2}\cdot\overset{3}{\cancel{3^3}}\cdot\overset{a^2b^3}{\cancel{a^3}\cancel{b^5}}\cancel{c^2}}{\underset{2^3}{\cancel{2^4}}\cdot\cancel{3^2}\cdot\cancel{a}\cancel{b^2}\underset{c^2}{\cancel{c^4}}}=\frac{3a^2b^3}{8c^2}.$$

2. $\frac{4a^8-324a^4}{6a^5-36a^3-162a}$.

Solution.
$$\frac{4a^8-324a^4}{6a^5-36a^3-162a}=\frac{\overset{2a^3}{\cancel{4}\cancel{a^4}}(a^2+9)\cancel{(a+3)}\cancel{(a-3)}}{\underset{3}{\cancel{6}\cancel{a}}(a^2+3)\cancel{(a+3)}\cancel{(a-3)}}$$
$$=\frac{2a^3(a^2+9)}{3(a^2+3)}.$$

The pupil should note that a factor which occurs one or more times in both numerator and denominator of a fraction can be canceled only the same number of times.

For reducing a fraction to its lowest terms we have the

Rule. *Separate the numerator and the denominator into their prime factors and cancel the factors common to both.*

Cancellation as used in the rule means an actual division of the numerator and the denominator by the same expression. Therefore *only factors which are common to the numerator and the denominator can be canceled.*

The terms (the parts connected by plus or minus signs) in polynomial numerators and denominators, even if alike, can never be canceled. For example, it would be incorrect to "cancel" thus: $\frac{5+\cancel{2}}{6+\cancel{2}}$, as the resulting fraction would be $\frac{5}{6}$ instead of the true value, $\frac{7}{8}$. Similarly, in the fraction $\frac{x+a+4c^2}{y+a+8c^2}$ no cancellation is possible.

We have seen that we may multiply or divide both numerator and denominator of a fraction by the same number without affecting the value of the fraction. But we should never forget that *adding the same number to or subtracting the same number from both numerator and denominator changes the value of the fraction. Also, squaring both numerator and denominator leads to a different value.* Compare this statement with the operations that may be performed on each member of an equation as given on pages 39–40.

EXERCISES

Reduce to lowest terms:

1. $\frac{36\,a}{54\,a^2}$.
2. $\frac{52\,x^2}{130\,xy}$.
3. $\frac{84\,m^2}{126\,mn}$.
4. $\frac{x^3y}{xy^3}$.
5. $\frac{xy^2}{2\,x^2y}$.
6. $\frac{5\,ab^2c}{3ac^2}$.
7. $\frac{10\,ax^2}{25\,ax^3}$.
8. $\frac{27\,cd^2e}{45\,c^2e^3}$.
9. $\frac{108\,m^2n^3}{252\,md^2}$.
10. $\frac{2\,a}{a^2+ab}$.
11. $\frac{4\,c^2-8\,cd}{12\,c}$.
12. $\frac{10\,x^2y^2-5\,xy^3}{15\,xy^3}$.
13. $\frac{9\,a^2-6\,a}{3\,a-2}$.
14. $\frac{4\,x^2-25}{4x^2-20\,x+25}$.
15. $\frac{9-6\,x+x^2}{9-x^2}$.
16. $\frac{9\,x^2-1}{9\,x^2-9\,x+2}$.
17. $\frac{10\,a^2-2\,a}{15\,a^2+7\,a-2}$.
18. $\frac{c^3-4\,c}{c^5-8\,c^2}$.
19. $\frac{c^2-6\,c+8}{c^2+c-6}$.
20. $\frac{2\,x^3+x^2-3\,x}{3\,x^4-3\,x^2}$.
21. $\frac{50\,x-2\,x^3}{x^3+8\,x^2+15\,x}$.
22. $\frac{x^3-49\,x}{42\,x^2-27\,x^3+3\,x^4}$.
23. $\frac{a^3-a^2+2\,a-2}{3\,a^3-a^2+6\,a-2}$.
24. $\frac{2\,a^3y+2\,a^2xy+2\,acy+2\,cxy}{2\,ax+2\,a^2-ay-xy}$.
25. $\frac{m^2-n^2}{m^3-n^3}$.
26. $\frac{x^4-8\,x}{x^3-2\,x^2}$.
27. $\frac{a^2-4}{(a-2)^2}$.
28. $\frac{a^6-1}{a^2-1}$.
29. $\frac{a^4-x^4}{a^4+3\,a^2x^2+2\,x^4}$.
30. $\frac{27\,x^3+a^3}{9\,x^2-a^2}$.

73. Lowest common multiple. The **lowest common multiple** (L.C.M.) of two or more arithmetical or algebraic expressions is the expression having the least number of factors which will exactly contain each of the given expressions.

If two or more polynomials have no common factor other than 1, they are said to be **prime** to each other.

Thus $5a^2b$ and $4x^2y$ are prime to each other, as also are $2x^2 + 4x$ and $x^2 - 9$. On the other hand, $x^2 - 9$ and $x^2 - 6x + 9$ are not prime to each other, since they contain the common factor $x - 3$.

ORAL EXERCISES

Without separating the expressions into their prime factors, find the L.C.M. of the following:

1. 4, 6.
2. 5, 10.
3. 6, 8.
4. 6, 9.
5. 8, 10.
6. 8, 12.
7. 7, 14.
8. 10, 15.
9. 12, 18.
10. 4, 6, 8.
11. 5, 10, 15.
12. 6, 10, 12.
13. 6, 8, 12.
14. 7, 14, 28.
15. 7, 14, 21.
16. 8, 10, 20.
17. 8, 20, 40.
18. 3, 12, 15.
19. 4, 8, 16.
20. 6, 18, 24.

EXAMPLE

Find the L.C.M. of $36a^2b^5$, $72a^3b^4$, and $108a^3b^2c^3$.

Solution.

$$36a^2b^5 = 2^2 \cdot 3^2 \cdot a^2b^5,$$
$$72a^3b^4 = 2^3 \cdot 3^2 \cdot a^3b^4,$$
$$108a^3b^2c^3 = 2^2 \cdot 3^3 \cdot a^3b^2c^3.$$

Since the L.C.M. must contain each of the expressions, it must have 2^3 as a factor. It will then contain 2^2, which occurs in both the first and the third monomials. Similarly, the L.C.M. must contain as factors 3^3, a^3, b^5, and c^3.

Therefore the L.C.M. is $2^3 \cdot 3^3 a^3b^5c^3$, which equals $216a^3b^5c^3$.

The method of finding the L.C.M. of two or more expressions is stated in the following

Rule. *Separate each expression into its prime factors. Then find the product of all the different prime factors, using each factor the greatest number of times it occurs in any one expression.*

EXERCISES

Find the L.C.M. of the following:

1. 96, 36, 40.

2. 45, 105, 175.

3. 50, 44, 110, 275.

4. a^2x, ax^3, a^2x^2.

5. mn, m^2, m^5n^2.

6. $2x, 6x^2, 4x^3$.

7. $3r, 12, 15r^2$.

8. $4a, 6a^2, 8a^5$.

9. $2ax^2, 3a^2x, a^2cx^3$.

10. $4x^2y, 6x^3yz^2, 10xz^2$.

11. $6a^2b, 9ab^2c, 30b^3c^3$.

12. $4c^2f, 10df^2, 24cd^2e^3$.

13. $2a^2c, 18ax^2k^3, 45x^3k$.

14. $x^2 - xy, 2x^2y^3$.

15. $a^2 + ab, 2ab$.

16. $5c - 10, 10c$.

17. $b^2 - bk, b^2k$.

18. $ac^2, 3a^2c,$
$6a^2c - 9ac^2$.

19. $a^2 + ab, ab + b^2$.

20. $3x^3 + 6x^2y + 3xy^2, 9x^3y - 9xy^3, 6x^4 - 12x^3y + 6x^2y^2$.

Solution. $3x^3 + 6x^2y + 3xy^2 = 3x(x+y)^2$

$$9x^3y - 9xy^3 = 3^2xy(x+y)(x-y)$$

$$6x^4 - 12x^3y + 6x^2y^2 = 2 \cdot 3x^2(x-y)^2.$$

Hence the L.C.M. is $2 \cdot 3^2x^2y(x+y)^2(x-y)^2$.

21. $a^2 - ab, 2a - 2b, 2a^2 - 2ab$.

22. $ax - ay, x^2 + xy, x^2 - y^2$.

23. $a^2 - 4, a^2 + 3a + 2, a^2 - a - 2$.

24. $4x^2 - 9, 2x^2 + 7x + 6, 2x^2 + x - 6$.

25. $x^2 - 1, x^3 - 1, x^2 + 2x + 1$.

26. $a^2 - 9, 2a^3 + 6a^2 + 18a, 2a^4 - 54a$.

27. $x^3 - 4x, 4x^2 + 2x, 2x^2 + 4x, 2x^2 - 3x - 2$.

74. Equivalent fractions. Two fractions are equivalent when one can be obtained from the other either by multiplying or by dividing both numerator and denominator by the same expression.

For example, $\frac{3}{5}$ and $\frac{6}{10}$ are equivalent fractions; also $\frac{xy}{x^2}$ and $\frac{y}{x}$.

The **lowest common denominator** (L.C.D.) of two or more fractions is the L.C.M. of their denominators.

EXAMPLES

Reduce to respectively equivalent fractions having the lowest common denominator :

1. $\frac{3}{5}$, $\frac{5}{6}$, and $\frac{7}{20}$.

Solution. The L.C.M. of the denominators is 60. Multiplying the numerator and the denominator of the first fraction by 12, of the second by 10, and of the third by 3, we obtain $\frac{36}{60}$, $\frac{50}{60}$, and $\frac{21}{60}$ respectively.

2. $\frac{3x}{8yz^2}$ and $\frac{5y}{6x^2z}$.

Solution. The L.C.M. of the denominators is $24x^2yz^2$. Multiplying both numerator and denominator of the first fraction by the factor $3x^2$, which is a factor of the L.C.M., but not of the denominator of the fraction, gives $\frac{9x^3}{24x^2yz^2}$. Multiplying both numerator and denominator of the second fraction by the factor $4yz$, which is a factor of the L.C.M., but not of the denominator of this fraction, gives $\frac{20y^2z}{24x^2yz^2}$. Hence the required fractions are $\frac{9x^3}{24x^2yz^2}$ and $\frac{20y^2z}{24x^2yz^2}$.

3. $\frac{4a}{9b^2c}$ and $\frac{8b}{6ac^2}$.

Solution. The L.C.D. is $18ab^2c^2$.

Then $\frac{4a}{9b^2c}=\frac{4a\cdot 2ac}{9b^2c\cdot 2ac}=\frac{8a^2c}{18ab^2c^2}$, and $\frac{8b}{6ac^2}=\frac{8b\cdot 3b^2}{6ac^2\cdot 3b^2}=\frac{24b^3}{18ab^2c^2}$

Therefore, to change two or more fractions (in their lowest terms) to respectively equivalent fractions having the L.C.D., we have the

Rule. *If the prime factors of the denominators are not apparent, rewrite the fractions with their denominators in factored form.*

Find the L.C.M. of the denominators of the fractions.

Multiply the numerator and the denominator of each fraction by those factors of this L.C.M. which are not found in the denominator of that fraction.

ORAL EXERCISES

State the lowest common denominator of the fractions in each of the following exercises. Then change the fractions to respectively equivalent fractions having this L.C.D.

1. $\frac{1}{2}, \frac{1}{3}$.
2. $\frac{1}{3}, \frac{1}{4}$.
3. $\frac{1}{3}, \frac{1}{6}$.
4. $\frac{1}{4}, \frac{1}{8}$.
5. $\frac{1}{6}, \frac{1}{10}$.
6. $\frac{1}{6}, \frac{1}{18}$.
7. $\frac{1}{10}, \frac{1}{15}$.
8. $\frac{1}{12}, \frac{1}{18}$.
9. $\frac{1}{15}, \frac{1}{20}$.
10. $\frac{1}{24}, \frac{1}{36}$.
11. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.
12. $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}$.
13. $\frac{1}{4}, \frac{1}{8}, \frac{1}{12}$.
14. $\frac{1}{10}, \frac{1}{15}, \frac{1}{20}$.
15. $\frac{1}{2}, \frac{1}{4}, \frac{3}{8}$.
16. $\frac{1}{2}, \frac{5}{6}, \frac{7}{12}$.
17. $\frac{1}{2}, \frac{3}{4}, \frac{3}{8}$.
18. $\frac{2}{3}, \frac{3}{4}, \frac{1}{6}$.
19. $\frac{2}{3}, \frac{5}{6}, \frac{7}{18}$.
20. $\frac{4}{5}, \frac{9}{10}, \frac{4}{15}$.

EXERCISES

Change the following fractions to respectively equivalent fractions having the lowest common denominator:

1. $\frac{x}{3}, \frac{y}{4}, \frac{z}{5}$.
2. $\frac{3x}{8}, \frac{5x^2}{12}, \frac{7x^3}{24}$.
3. $\frac{5a}{7}, \frac{3ax}{14}, \frac{2ax^2}{21}$.
4. $\frac{3a^2}{4}, \frac{6ab}{10}, \frac{15a^2b}{24}$.
5. $\frac{3}{5a}, \frac{2c}{3d}, \frac{5a}{10d}$.
6. $\frac{3x}{4y}, \frac{5x}{6y}, \frac{3}{10y}$.
7. $\frac{2a+b}{3a}, \frac{a}{2a^2}, \frac{5a^2}{4a}$.
8. $\frac{c+d}{12c}, \frac{3c-d}{15c^2}, \frac{4a}{24c^3}$.
9. $\frac{7a}{cde^2}, \frac{5e}{cd^2e}, \frac{4c}{c^2d^2e^2}$.
10. $\frac{7x}{3xy^2}, \frac{4y}{5xz}, \frac{2z}{10yz^2}$.

11. $\frac{3}{x-5}, \frac{4}{x+5}.$

Solution. The L.C.D. is $(x-5)(x+5)$. Multiplying both numerator and denominator of the first fraction by $x+5$, and those of the second fraction by $x-5$, gives $\frac{3(x+5)}{(x-5)(x+5)}$ and $\frac{4(x-5)}{(x+5)(x-5)}$, or $\frac{3x+15}{x^2-25}$ and $\frac{4x-20}{x^2-25}$ respectively.

12. $\frac{5}{a-2}, \frac{7}{a+2}.$

13. $\frac{a}{a-b}, \frac{b}{a+b}.$

14. $\frac{x}{2x-5}, \frac{3x}{2x+5}.$

15. $\frac{-2c}{c-3}, \frac{3c}{c+3}.$

16. $\frac{2a}{4-a}, \frac{a}{4+a}, \frac{2}{a}.$

Solution. The L.C.D. is $(4-a)(4+a)a$. The respectively equivalent fractions are

$$\frac{2a(4+a)a}{(4-a)(4+a)a}, \frac{a(4-a)a}{(4+a)(4-a)a}, \text{ and } \frac{2(4-a)(4+a)}{a(4-a)(4+a)}.$$

17. $\frac{x}{x+y}, \frac{2x}{x-y}, \frac{2}{x}.$

18. $\frac{2a}{a-3}, \frac{3}{a+3}, \frac{-a}{2}.$

19. $\frac{5c}{c-5}, \frac{-2c}{c-5}, \frac{3}{c}.$

20. $\frac{m}{m-n}, \frac{-2m}{m-n}, \frac{2}{m}.$

21. $\frac{5}{a-2}, \frac{2}{3a-6}.$

HINT. Rewrite the fractions with the second denominator in factored form.

22. $\frac{a}{x^2-xy}, \frac{5}{x-y}.$

23. $\frac{3a}{a^2-b^2}, \frac{2a}{a+b}.$

24. $\frac{5a}{a^2-b^2}, \frac{-4a}{a-b}.$

25. $\frac{-3x}{x^2-y^2}, \frac{5x}{x-y}.$

26. $\frac{2x+3}{9-x^2}, \frac{3-2x}{9+3x}, \frac{x}{3-x}.$

27. $\frac{-2x}{x-2}, \frac{3}{x^2-4}, \frac{5x}{x^2-5x+6}.$

28. $\frac{2a-7}{a^3-9a}, \frac{3-5a}{a^3-5a^2+6a}, \frac{2}{a-3}.$

29. $\frac{a}{a^3-8}, \frac{2a}{a^3+2a^2+4a}.$

30. $\dfrac{5x}{2x^2+3x}, \dfrac{3x}{2x^2-x-6}.$

31. $\dfrac{-4x}{3-x}, \dfrac{5x}{3+8x-3x^2}.$

32. $\dfrac{-1}{x+y-3}, \dfrac{5x+y}{x^2+y^2-9+2xy}.$

NOTE. The problem of operating with fractions presented great difficulties to all the early races. The Egyptians and the Greeks, even down to the sixth century of our era, always reduced their fractions to the sum of several fractions, each of which had 1 for a numerator. For example, $\frac{5}{8}$ would be expressed as $\frac{1}{2}+\frac{1}{8}$. The Romans usually expressed all the fractions of a sum in terms of fractions with the common denominator 12. The Babylonians resorted to a similar device, but used 60 for the denominator. In some way they all attempted to evade the difficulty of considering changes in both numerator and denominator. The Hindus seem to have been the first to reduce fractions to a common denominator, though Euclid (300 B.C.) was familiar with the method of finding the least common multiple of two or more numbers.

75. Addition and subtraction of fractions. If two or more fractions have the same denominator, their sum is the fraction obtained by adding their numerators and writing the result over their common denominator.

For example, $\frac{2}{5}+\frac{4}{5}+\frac{7}{5}=\frac{13}{5}$, and $\frac{n}{d}+\frac{3n}{d}+\frac{5n}{d}=\frac{9n}{d}$.

If two fractions have the same denominators, their difference is the fraction obtained by subtracting the numerator of the subtrahend from the numerator of the minuend and writing the result over their common denominator.

For example, $\frac{7}{9}-\frac{5}{9}=\frac{2}{9}$, and $\frac{x}{d}-\frac{y}{d}=\frac{x-y}{d}$.

If it is required to add or to subtract two fractions having unlike denominators, the fractions must be changed to respectively equivalent fractions having a common denominator; then their sum or their difference is obtained as explained above.

For example, to find the sum of $\frac{3}{4}+\frac{5}{8}+\frac{7}{10}$ we reduce the fractions to respectively equivalent fractions having the common denominator 40, by multiplying both numerator and denominator of $\frac{3}{4}$ by 10, of $\frac{5}{8}$ by 5, and of $\frac{7}{10}$ by 4. The fractions become $\frac{30}{40}$, $\frac{25}{40}$, and $\frac{28}{40}$ respectively, and their sum is $\frac{83}{40}$.

In adding algebraic fractions with unlike denominators, as $\frac{x}{y}$ and $\frac{n}{d}$, we proceed in a similar way.

Multiply both terms of $\frac{x}{y}$ by d, and of $\frac{n}{d}$ by y. The fractions become $\frac{xd}{yd}$ and $\frac{ny}{yd}$ respectively, the sum of which is $\frac{xd+ny}{yd}$. Similarly, $\frac{x}{y}-\frac{n}{d}=\frac{xd}{yd}-\frac{ny}{yd}$, which equals $\frac{xd-ny}{yd}$.

The pupil should always reduce fractional results to their lowest terms.

ORAL EXERCISES

Find the algebraic sum of:

1. $\frac{2}{3}+\frac{5}{3}$.
2. $\frac{7}{5}+\frac{8}{5}$.
3. $\frac{8}{9}+\frac{11}{9}$.
4. $\frac{13}{7}+\frac{6}{7}$.
5. $\frac{11}{5}+\frac{7}{5}$.
6. $\frac{14}{8}+\frac{11}{3}$.
7. $\frac{12}{15}+\frac{10}{15}$.
8. $\frac{17}{9}-\frac{1}{9}$.
9. $\frac{21}{a}-\frac{2}{a}$.
10. $\frac{3x}{y}-\frac{x}{y}$.
11. $\frac{a}{b}-\frac{c}{b}$.
12. $\frac{m}{n}-\frac{2m}{n}$.
13. $\frac{a}{2x}-\frac{c}{2x}$.
14. $\frac{5x}{y^3}-\frac{2a}{y^3}$.
15. $\frac{3x}{4x^2}+\frac{5}{4x^2}$.
16. $\frac{2a+c}{xy}-\frac{a}{xy}$.
17. $\frac{3r-s}{2r}-\frac{s-r}{2r}$.
18. $\frac{2x-y}{3x}-\frac{2y+x}{3x}$.
19. $\frac{x^2+y^2}{x+y}-\frac{y^2-x^2}{x+y}$.
20. $\frac{3x+5y}{x^2-y^2}-\frac{x-2y}{x^2-y^2}$.
21. $\frac{3}{a^2-b^2}-\frac{2}{a^2-b^2}$.

EXAMPLE

Simplify $\frac{5c}{2a^2}+\frac{5a-2c}{6ac}-\frac{2c-4}{3a}$.

Solution. The L.C.D. is $6a^2c$.

$$\begin{aligned}\frac{5c}{2a^2}+\frac{5a-2c}{6ac}-\frac{2c-4}{3a}&=\frac{5c\cdot 3c}{2a^2\cdot 3c}+\frac{(5a-2c)a}{6ac\cdot a}-\frac{(2c-4)2ac}{3a\cdot 2ac}\\&=\frac{15c^2+(5a^2-2ac)-(4ac^2-8ac)}{6a^2c}\\&=\frac{15c^2+5a^2-2ac-4ac^2+8ac}{6a^2c}\\&=\frac{5a^2+6ac-4ac^2+15c^2}{6a^2c}.\end{aligned}$$

Check. Setting the original expression equal to the final result and substituting 2 for a and 3 for c, we obtain

$$\begin{aligned}\frac{5c}{2a^2}+\frac{5a-2c}{6ac}-\frac{2c-4}{3a}&=\frac{5a^2+6ac-4ac^2+15c^2}{6a^2c}.\\ \frac{15}{8}+\frac{1}{9}-\frac{1}{3}&=\frac{20+36-72+135}{72}.\\ \frac{135}{72}+\frac{8}{72}-\frac{24}{72}&=\frac{119}{72}.\\ \frac{119}{72}&=\frac{119}{72}.\end{aligned}$$

Therefore, to find the algebraic sum of two or more fractions (in their lowest terms) we have the

Rule. *Reduce the fractions to respectively equivalent fractions having the lowest common denominator. Write in succession over the lowest common denominator the numerators of the equivalent fractions, inclosing each numerator in a parenthesis preceded by the sign of the corresponding fraction.*

Rewrite the fraction just obtained, removing the parentheses in the numerator.

Then combine like terms in the numerator and, if necessary, reduce the resulting fraction to its lowest terms.

EXERCISES

Find the algebraic sum of:

1. $\frac{3}{4}+\frac{5}{6}$.
2. $\frac{3}{10}+\frac{8}{15}$.
3. $\frac{9}{25}-\frac{3}{10}$.
4. $\frac{5}{12}-\frac{7}{18}$.
5. $\frac{3}{14}+\frac{2}{21}$.
6. $\frac{5}{6}-\frac{3}{16}$.
7. $\frac{7}{24}-\frac{15}{32}$.
8. $\frac{17}{40}+\frac{19}{48}$.
9. $\frac{15}{28}-\frac{87}{40}$.
10. $\frac{43}{25}+2\frac{29}{45}$.
11. $\frac{19}{24}-\frac{89}{64}$.
12. $\frac{55}{38}+\frac{43}{57}$.
13. $\frac{7}{6}-1\frac{4}{9}+3\frac{7}{12}$.
14. $\frac{11}{10}-\frac{13}{12}+2\frac{5}{16}$.
15. $\frac{13}{32}+1-3\frac{9}{48}$.
16. $\frac{31}{36}-\frac{41}{32}+\frac{11}{42}$.
17. $\frac{15}{26}-\frac{11}{39}-\frac{25}{52}$.
18. $\frac{23}{35}-\frac{19}{45}+\frac{27}{50}$.
19. $\frac{19}{48}-\frac{13}{72}+\frac{43}{36}$.
20. $\frac{11}{64}+2-\frac{17}{24}$.
21. $\frac{14}{69}-\frac{15}{46}+\frac{19}{115}$.

22. $\frac{3a}{2}+\frac{4a}{3}$.

23. $\frac{2a}{5}+\frac{3a}{2}+\frac{11a}{10}$.

24. $\frac{7a}{10}-\frac{6a}{15}+\frac{3a}{5}$.

25. $\frac{4x}{7}+\frac{2x}{21}-\frac{3x}{14}$.

26. $\frac{a+2}{8}-\frac{3a-2}{10}$.

27. $\frac{5x-3}{12}-\frac{2x-7}{14}$.

28. $\frac{3a-5}{4}-\frac{2-a}{6}+\frac{7a}{8}$.

29. $\frac{x-5a}{10}-\frac{3x+2a}{18}-\frac{a-7x}{20}$.

30. $\frac{4m-1}{6}-\frac{a-m}{8}-\frac{m+3a-5}{30}$.

31. $\frac{3}{x}+\frac{2}{3x^2}$.

32. $\frac{2}{3a}+\frac{5}{2a^2}-\frac{10a}{4a^3}$.

33. $\frac{2a}{5a^3}-\frac{3}{10a}-\frac{5}{6a^2}$.

34. $\frac{a}{x}+\frac{x}{a}-\frac{2}{ax}$.

35. $\frac{a}{c}-\frac{1}{ac}-\frac{3}{2a}$.

36. $\frac{7}{3a^2}-\frac{2}{ax}+\frac{3}{x^2}$.

37. $\frac{2a}{3x}+\frac{11}{2ax^2}-\frac{5x}{a}$.

38. $\frac{3a+1}{5a}-\frac{4a+3}{3a^2}-\frac{5-a}{6a}$.

39. $\frac{4x^2-5}{3x^2}-\frac{2-3x}{2x}+\frac{3x-7}{5x^3}$.

40. $\frac{2x}{5x^2y}-\frac{3y-1}{10xy^2}-\frac{4xy+3}{15x^2y^2}$.

41. $\frac{3a+2}{a^2-4}-\frac{2a-1}{a^2-3a-10}$.

Solution.

$$\frac{3a+2}{a^2-4}-\frac{2a-1}{a^2-3a-10}=\frac{3a+2}{(a+2)(a-2)}-\frac{2a-1}{(a+2)(a-5)}$$
$$=\frac{(3a+2)(a-5)}{(a+2)(a-2)(a-5)}-\frac{(2a-1)(a-2)}{(a+2)(a-5)(a-2)}$$
$$=\frac{(3a+2)(a-5)-(2a-1)(a-2)}{(a+2)(a-2)(a-5)}$$
$$=\frac{3a^2-13a-10-(2a^2-5a+2)}{(a+2)(a-2)(a-5)}$$
$$=\frac{3a^2-13a-10-2a^2+5a-2}{(a+2)(a-2)(a-5)}$$
$$=\frac{a^2-8a-12}{(a+2)(a-2)(a-5)} \text{ or } \frac{a^2-8a-12}{a^3-5a^2-4a+20}.$$

Unless other directions are given, the denominators should be retained in factored form throughout the process.

Check. Let $a=3$

$$\frac{3a+2}{a^2-4}-\frac{2a-1}{a^2-3a-10}=\frac{a^2-8a-12}{(a+2)(a-2)(a-5)}.$$
$$\frac{9+2}{9-4}-\frac{6-1}{9-9-10}=\frac{9-24-12}{5\cdot1\cdot(-2)}.$$
$$\frac{11}{5}-\frac{5}{-10}=\frac{-27}{-10} \text{ or } \frac{27}{10}=\frac{27}{10}.$$

In checking work in fractions *such values must be chosen for the letters as will make no denominator zero.* This prevents the substitution of -2, 2, or 5 for a in the foregoing example.

42. $\frac{12}{x^2-9}-\frac{2}{x^2-5x+6}$.

43. $\frac{7}{a^2-49}-\frac{2}{a^2-6a-7}$.

44. $\frac{5}{4x^2-1}-\frac{3x}{8x^3-1}$.

45. $\frac{4c+3}{1-c^2}+\frac{2c}{c+c^2}$.

46. $\frac{2y+7}{3+y}-\frac{2y^2-35}{y^2-11y-42}$.

47. $\frac{5a+1}{a+3}-\frac{7}{2a}+\frac{2}{3}$.

48. $\frac{2}{x^2-7x}-\frac{3}{x}+\frac{3}{x-7}$.

49. $\frac{7a+b}{a^2-b^2}+\frac{2}{a-b}-\frac{3}{a+b}$.

50. $\frac{5c^2}{4c^2-1}-\frac{3c+2}{2c+1}+\frac{5c}{2c-1}$.

51. $\frac{m^3+m}{m^3-8}-\frac{3m+7}{m^3+2m^2+4m}-\frac{m-3}{m-2}$.

52. $\frac{a^2+2ab+b^2}{a^2-b^2}+\frac{ab}{a^2+ab}-\frac{2b}{ab-b^2}$.

53. $\frac{x^2+3}{x^2-x-2}-\frac{x}{x-2}+\frac{x}{x^2+x}$.

54. $\frac{a}{a^2-ab}+\frac{b+a}{a^2-2ab+b^2}+\frac{2b}{-ab+a^2}$.

55. $\frac{cx^2}{c^2-x^2}-\frac{2cx}{c^2-cx}-\frac{cx^2(c-2)}{c^3-cx^2}$.

56. $\frac{a+c}{a-c}-\frac{a^2-ac+c^2}{a^2+ac+c^2}-\frac{4a^2c}{a^3-c^3}$.

57. $\frac{x^2+3x+9}{x^2-3x+9}-\frac{x-3}{x+3}-\frac{54}{x^3+27}$.

58. $\frac{x-5}{2x+5}+\frac{-5x+x^2}{2x^2+15x+25}-\frac{2x-5}{x+5}$.

76. Changes of sign in a fraction. The *sign of a fraction* is the plus or minus sign placed before the line separating the numerator from the denominator. Hence there are in a fraction three signs to consider, — the sign of the fraction, the sign of the numerator, and the sign of the denominator.

Now in division the quotient of two expressions having like signs is positive, and the quotient of two expressions having unlike signs is negative.

Therefore $+\frac{+8}{+2}=+4;\qquad +\frac{-8}{-2}=+4;$

$$-\frac{-8}{+2}=-(-4)=+4;\qquad -\frac{+8}{-2}=-(-4)=+4.$$

Or, in general terms, $+\frac{+a}{+b}=+\frac{-a}{-b}=-\frac{-a}{+b}=-\frac{+a}{-b}$.

These examples illustrate the

Principle. *Without altering the value of a fraction the following changes in sign may be made:*

(a) The sign of the numerator and the sign of the denominator.

(b) The sign of the numerator and the sign before the fraction.

(c) The sign of the denominator and the sign before the fraction.

Hence any fraction may be written in at least four ways, if proper changes of sign are made.

Thus $$+\frac{3x}{x-5}=\frac{-3x}{5-x}=-\frac{-3x}{x-5}=-\frac{3x}{5-x}.$$

Similarly,

$$+\frac{a-2b}{a-b+3c}=+\frac{2b-a}{b-a-3c}=-\frac{2b-a}{a-b+3c}=-\frac{a-2b}{b-a-3c}.$$

The pupil should note particularly that changing the sign of the numerator involves a change of sign in each term of the numerator. Similarly, a change of sign of the denominator involves a change of sign in each term of the denominator.

EXERCISES

Write each of the following fractions in three other ways:

1. $\frac{a}{-h}$.
2. $\frac{-c}{2d}$.
3. $-\frac{n}{d}$.
4. $-\frac{2}{a-c}$.
5. $\frac{-3}{x-y}$.
6. $\frac{x-2}{2x-3}$.
7. $-\frac{2a-b}{b^2-a^2}$.
8. $\frac{c-3d}{d^2-c^2}$.
9. $\frac{x-5}{5x-6-x^2}$.
10. $-\frac{2-x}{1-x^3}$.

77. Changing signs of factors of denominators. In their present form the L.C.D. of $\frac{a-3}{a-7}-\frac{2}{7-a}$ is apparently $(a-7)(7-a)$. But instead of taking these factors it is better to apply the principle of section 76 and rewrite the fractions so that the denominators will be identical.

Thus $$\frac{a-3}{a-7}-\frac{2}{7-a}=\frac{a-3}{a-7}+\frac{2}{a-7}=\frac{a-1}{a-7}.$$

And $$\frac{x}{x-3}+\frac{4-2x}{3-x}=\frac{x}{x-3}+\frac{2x-4}{x-3}=\frac{3x-4}{x-3}.$$

Similarly, $$\frac{3}{x^2-4}-\frac{5}{2-x}=\frac{3}{(x+2)(x-2)}+\frac{5}{x-2}\text{ etc.}$$

In addition and subtraction of fractions whenever a factor occurs in one denominator which is the negative of a factor in another, the form of one of the fractions should be changed according to section 76. This will greatly simplify the work and decrease the likelihood of errors.

EXERCISES

Simplify the following:

1. $\dfrac{7}{x-5}+\dfrac{3}{5-x}.$

2. $\dfrac{a}{a-b}-\dfrac{b}{b-a}.$

3. $\dfrac{x^2-2a}{x-a}+\dfrac{a+x^2}{a-x}.$

4. $\dfrac{x-2}{3-x}-\dfrac{2+x}{x-3}+\dfrac{3x}{x-3}.$

5. $\dfrac{a}{5-a}+\dfrac{a^2+5a}{a^2-25}.$

Hint. $$\frac{a}{5-a}+\frac{a^2+5a}{a^2-25}=\frac{-a}{a-5}+\frac{a^2+5a}{(a+5)(a-5)}\text{ etc.}$$

6. $\dfrac{2}{c^2-9}-\dfrac{9}{3-c}.$

7. $\dfrac{3}{7-x}-\dfrac{4}{x^2-49}.$

8. $\dfrac{2a+5}{5a+1}+\dfrac{2a-5}{1-5a}-\dfrac{46a+1}{25a^2-1}.$

9. $\dfrac{a-b}{3a+2b}-\dfrac{10ab}{4b^2-9a^2}-\dfrac{a+b}{3a-2b}.$

10. $\dfrac{2x+1}{2x-1}-\dfrac{-6x+1}{1-4x^2}-\dfrac{2x-1}{2x+1}.$

11. $\dfrac{-11a+46-6a^2}{a^2+5a-14}+\dfrac{2a-5}{a+7}+\dfrac{7-4a}{2-a}.$

78. Reduction of a mixed expression to a fraction. The mixed number $3\frac{2}{5}$ really means $3+\frac{2}{5}$. It is equal to $\frac{3}{1}+\frac{2}{5}$ or $\frac{15}{5}+\frac{2}{5}=\frac{17}{5}$. The corresponding case in algebra is represented by $a+\frac{b}{c}$.

Now $$a+\frac{b}{c}=\frac{a}{1}+\frac{b}{c}=\frac{ac}{c}+\frac{b}{c}=\frac{ac+b}{c}.$$

Similarly,

$$a+1+\frac{3a}{a+2}=\frac{a+1}{1}+\frac{3a}{a+2}=\frac{(a+1)(a+2)}{a+2}+\frac{3a}{a+2}=\frac{a^2+6a+2}{a+2}.$$

The process involved in this operation is really nothing more than the addition of two fractions, one of which has the denominator 1. Hence the method of section 75 applies.

ORAL EXERCISES

Reduce to fractional form:

1. $2+\frac{1}{5}$.
2. $3+\frac{1}{3}$.
3. $4+\frac{1}{7}$.
4. $5+\frac{2}{3}$.
5. $6+\frac{5}{7}$.
6. $\frac{2}{3}+3$.
7. $\frac{7}{8}+3$.
8. $\frac{11}{2}-4$.
9. $\frac{13}{5}-3$.
10. $2-\frac{4}{5}$.
11. $7-\frac{6}{7}$.
12. $5+\frac{1}{3}$.
13. $7-\frac{3}{4}$.
14. $12+\frac{2}{9}$.
15. $1+\frac{a}{b}$.
16. $2-\frac{a}{c}$.
17. $a-\frac{a}{x}$.
18. $c-\frac{x}{c}$.
19. $r-\frac{s^2}{r}$.
20. $s^2-\frac{1}{4}$.
21. $s^2-\frac{s^2}{4}$.
22. $r^2-\left(\frac{r}{2}\right)^2$.
23. $\frac{s^2}{4}+s^2$.
24. $\left(\frac{s}{2}\right)^2-s^2$.
25. $\left(\frac{r}{3}\right)^2-\frac{-r^2}{9}$.
26. $\left(\frac{r}{5}\right)^2-\frac{r^2}{5}$.
27. $2+\frac{3}{r+1}$.
28. $5-\frac{2}{s-1}$.
29. $x+\frac{1}{x+1}$.
30. $y-\frac{2}{y-2}$.

EXERCISES

Write the following mixed expressions in fractional form:

1. $a + \frac{2}{x}$.

2. $b - \frac{c}{d}$.

3. $2c + \frac{c}{a}$.

4. $5x - \frac{x}{d}$.

5. $\frac{m}{n} - 2$.

6. $\frac{c}{5d} + 3$.

7. $s + \frac{s}{2}$.

8. $x^2 - \frac{3x^2}{5}$.

9. $s^2 - \left(\frac{s}{2}\right)^2$.

10. $x^2 - \left(\frac{2x}{3}\right)^2$.

11. $2a^2 - \frac{2a^2}{3}$.

12. $a + 2 + \frac{a-2}{2}$.

13. $c - d + \frac{c^2 - d^2}{c + d}$.

14. $x - 2y - \frac{x^2 + y^2}{x + 2y}$.

15. $a^2 + b^2 - \frac{(a+b)^2}{2}$.

16. $5a - 7 - \frac{25a^2}{5a + 7}$.

17. $m^2 + mn + n^2 - \frac{2m^3 - n^3}{m - n}$.

18. $x^2 - xy - \frac{x^3 - y^3}{x + y} + y^2$.

19. $5 - \frac{125 - 2x^3}{25 + 5x + x^2} - x$.

20. $\frac{x+2}{x-2} - 3 + \frac{x-3}{2x+3}$.

21. $x^3 + 4x - \frac{x^4 - 18}{x - 2} + 8 + 2x^2$.

22. $b + 2 - \frac{14b}{2b-1} - \left(2b - 1 - \frac{b^2 + 9b}{b + 5}\right)$.

HINT. Removing parentheses and combining, we get

$$b + 2 - \frac{14b}{2b-1} - 2b + 1 + \frac{b^2+9b}{b+5} = 3 - b - \frac{14b}{2b-1} + \frac{b^2+9b}{b+5} \text{ etc.}$$

23. $\left(2 + \frac{6a}{b}\right) - \left(3 - \frac{4a}{3b}\right)$.

24. $\left(3 - \frac{7}{2ac}\right) - \left(2 - \frac{5}{3c^2}\right)$.

25. $\left(x + \frac{4y^2}{x - 2y}\right) - \left(\frac{x^2}{x + 2y} + 2y\right)$.

79. Multiplication of fractions. In algebra as in arithmetic the product of two or more fractions is the product of their numerators divided by the product of their denominators.

Thus $\frac{3}{4} \cdot \frac{5}{7} = \frac{15}{28}$.

Similarly, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$,

and $5 \cdot \frac{3}{7} = \frac{5}{1} \cdot \frac{3}{7} = \frac{15}{7}$.

In like manner $n \cdot \frac{a}{b} = \frac{n}{1} \cdot \frac{a}{b} = \frac{na}{b}$.

If a factor occurs one or more times in any numerator and in any denominator of the indicated product of two or more fractions, it should be canceled the same number of times from both, thus giving the product of the several fractions in lower terms.

Thus $\frac{a}{\cancel{b}} \cdot \frac{\cancel{b}}{c} = \frac{a}{c}$.

EXAMPLE

Multiply $\frac{8\,ac^2}{9\,d}$ by $\frac{15\,d^2e}{16\,a^2c^4}$.

Solution. $\frac{8\,ac^2}{9\,d} \cdot \frac{15\,d^2e}{16\,a^2c^4} = \frac{\cancel{8}\,\cancel{a}\,\cancel{c^2}}{\underset{3}{\cancel{9}\,\cancel{d}}} \cdot \frac{\overset{5\,d}{\cancel{15}\,\cancel{d^2}}\,e}{\underset{2\,a\,c^2}{\cancel{16}\,\cancel{a^2}\,\cancel{c^4}}} = \frac{5\,de}{6\,ac^2}$.

To find the product of two or more fractions or mixed expressions we have the

Rule. *If there are integral or mixed expressions, reduce them to fractional form.*

Separate each numerator and each denominator into its prime factors.

Cancel the factors common to any numerator and any denominator.

Write the product of the factors remaining in the numerator over the product of the factors remaining in the denominator.

ORAL EXERCISES

Simplify:

1. $\frac{1}{2}\cdot\frac{1}{3}$.
2. $\frac{1}{5}\cdot\frac{1}{4}$.
3. $\frac{2}{5}\cdot\frac{1}{3}$.
4. $\frac{3}{7}\cdot\frac{7}{2}$.
5. $\frac{5}{6}\cdot\frac{12}{7}$.
6. $\frac{8}{11}\cdot\frac{22}{10}$.
7. $(\frac{9}{10})^2$.
8. $\frac{a}{d}\cdot\frac{d}{c}$.
9. $\frac{c}{x}\cdot\frac{x}{c}$.
10. $\frac{m}{n}\cdot\frac{n^2}{m^2}$.
11. $\frac{x^2}{y^2}\cdot\frac{y}{x}$.
12. $\frac{x^2}{y^3}\cdot\frac{y}{x}$.
13. $\left(\frac{c^2}{d}\right)^2$.
14. $\frac{2x}{5}\cdot\frac{10}{3x}$.
15. $\left(\frac{5s^2}{t}\right)^2$.
16. $\left(\frac{2a}{3b^2}\right)^3$.
17. $\left(\frac{x+y}{2x^2}\right)^2$.
18. $\frac{m-n}{mn}\cdot\frac{m}{m+n}$.
19. $\frac{2x+c}{xc}\cdot\frac{xc^2}{x-2c}$.

EXERCISES

Multiply:

1. $\frac{12}{15}\cdot\frac{10}{20}$.
2. $\frac{4}{21}\cdot\frac{12}{10}$.
3. $\frac{4}{9}\cdot\frac{15}{22}\cdot 3$.
4. $1\frac{1}{6}\cdot\frac{8}{21}$.
5. $-\frac{14}{11}\cdot\frac{33}{42}$.
6. $2\frac{4}{7}\cdot -1\frac{17}{18}$.
7. $2\frac{1}{4}\cdot\frac{1}{21}\cdot 1\frac{7}{8}$.
8. $-1\frac{1}{11}\cdot 1\frac{11}{21}$.
9. $\frac{14}{24}\cdot 16\cdot\frac{32}{70}$.
10. $\frac{22}{7}\cdot(6)^2\cdot\frac{1}{144}$.
11. $\frac{4}{3}\cdot\frac{22}{7}\cdot(2\frac{1}{2})^3$.
12. $\frac{5a^2b}{3c^5}\cdot\frac{6c^2}{10ab^3}$.
13. $\frac{10xy^2}{18z^2}\cdot\frac{24z^3}{15x^4}$.
14. $\frac{15ax^3}{4x}\cdot\frac{12a^5}{40x^2}\cdot\frac{16x^2}{9a^5}$.
15. $\frac{8cd^2}{3x}\cdot\frac{18}{40c^3d^5}\cdot 5x^2d$.
16. $\left(\frac{5x}{2ac}\right)^2\cdot\frac{6a^2}{10x^3}\cdot\frac{5c^5}{3a^4}$.
17. $21x^2y\cdot\frac{9ay^2}{14x^3}\left(\frac{2x}{3y^2}\right)^3$.
18. $\left(\frac{a}{5c^2}\right)^2\cdot\left(\frac{-c}{2d}\right)^3\cdot\frac{15d}{ac}$.
19. $\frac{10m}{n^7}\cdot\left(\frac{4n^2}{15m}\right)^2\cdot\left(\frac{5m^3}{6m^2n}\right)^2$.
20. $\frac{(2x)^2}{3y}\cdot\frac{(3y)^2}{2x^2}$.
21. $\frac{(6a)^2}{(10b^2)^3}\cdot\frac{125b^4}{2(a^3)^2}$.
22. $\left(\frac{8m^2}{10mn^2}\right)^2\cdot\left(\frac{15n^3}{4m}\right)^3$.
23. $\frac{a^2-9}{a^2-2a-15}\cdot\frac{a-4}{2a^3-8a^2}\cdot\frac{2a^3-10a^2}{a^2-6a+9}$.

Solution. $\frac{a^2-9}{a^2-2a-15}\cdot\frac{a-4}{2a^3-8a^2}\cdot\frac{2a^3-10a^2}{a^2-6a+9}=$

$$\frac{\cancel{(a-3)}\cancel{(a+3)}}{\cancel{(a+3)}\cancel{(a-5)}}\cdot\frac{\cancel{a-4}}{\cancel{2a^2}\cancel{(a-4)}}\cdot\frac{\cancel{2a^2}\cancel{(a-5)}}{\cancel{(a-3)}(a-3)}=$$

$$\frac{1}{a-3}.$$

Check. Let $a=2$.

$$\frac{a^2-9}{a^2-2a-15}\cdot\frac{a-4}{2a^3-8a^2}\cdot\frac{2a^3-10a^2}{a^2-6a+9}=\frac{1}{a-3}.$$

$$\frac{-5}{-15}\cdot\frac{-2}{-16}\cdot\frac{-24}{1}=\frac{1}{-1}.$$

$$\frac{1}{3}\cdot\frac{1}{8}\cdot-24=-1.$$

$$-1=-1.$$

24. $\frac{2x^2+6}{5y^3}\cdot\frac{10y^4}{3x^2+9}.$

25. $\frac{c^2-4}{x-y}\cdot\frac{3x-3y}{c^2+6c+8}.$

26. $\frac{x^2-2x+4}{x^2-9}\cdot\frac{x^2-x-6}{x^3+8}.$

27. $\frac{x^2-3x+9}{2x-6}\cdot\frac{x^2-9}{x^3+27}.$

28. $\frac{x^2-7x+10}{x^3-125}\cdot\frac{2x^3+10x^2+50x}{x^2-4}.$

29. $\frac{x^4-16}{(x^2-4)^2}\cdot\frac{x^2-4}{4+x^2}.$

30. $\frac{c^2-9}{a^2-4x^2}\cdot\frac{a^2+4ax+4x^2}{ac+2cx-3a-6x}.$

31. $\frac{2a^2-5ac-3c^2}{c^2-9a^2}\cdot\frac{3c+9a}{10a^2+5ac}.$

32. $\frac{2x^2+10x+50}{3x^3-12x}\cdot\frac{x^2-3x-10}{4x^3-500}\cdot\frac{2-x}{2}.$

33. $\frac{(x^2-4)(x+3)}{2x^2+12x+18}\cdot\frac{x^2-x-6}{(x^2-4)^2}\cdot\frac{(x-2)}{x+3}.$

34. $\frac{4e^2-12ex+9x^2}{(e^2+ex+x^2)(3x-2e)}\cdot(e^3-x^3)\cdot\frac{x-e}{2e^2-5ex+3x^2}.$

35. $\left(4+\dfrac{3}{a^2-1}\right)\left(\dfrac{3a}{2a-1}-1\right)$.

HINT. $\left(4+\dfrac{3}{a^2-1}\right)\left(\dfrac{3a}{2a-1}-1\right)=\dfrac{4a^2-1}{a^2-1}\cdot\dfrac{a+1}{2a-1}$ etc.

The student should note particularly the difference between the procedure in multiplying two expressions like the above and that followed in adding them or in subtracting one from the other. (See Exercise 22, p. 166.)

36. $\left(\dfrac{x^2}{y^2}-4\right)\left(\dfrac{2x+4y}{x^3-8y^3}\right)\left(\dfrac{x^2+2xy+4y^2}{x^2+4xy+4y^2}\right)$.

37. $\left(3x-5-\dfrac{168}{9x+15}\right)\left(3-\dfrac{4}{3+x}\right)$.

38. $\left(3x-4+\dfrac{21}{4+x}\right)\left(\dfrac{1}{x+1}\right)^2\left(x+\dfrac{x-3}{3x+5}+3\right)$.

39. $\left(2x-\dfrac{1}{2x}\right)\left(6x+\dfrac{3}{2x+1}\right)\left(\dfrac{1}{8}\right)\dfrac{1}{8x^3-1}$.

40. $\left(\dfrac{4}{x^2}-\dfrac{5}{x}+1\right)\left(\dfrac{5x^4+5x^3}{x^2-11x+28}\right)\left(1-\dfrac{7x-1}{x^2-1}\right)$.

41. $\left(2b+\dfrac{a^2}{2b}-2a\right)\left(a+\dfrac{12b^2}{a-2b}+4b\right)\cdot\dfrac{2b^2}{a^3-8b^3}$.

80. Division of fractions. In arithmetic $\frac{3}{4}\div\frac{5}{7}=\frac{3}{4}\cdot\frac{7}{5}=\frac{21}{20}$; and $\frac{3}{4}\div 11=\frac{3}{4}\cdot\frac{1}{11}=\frac{3}{44}$. Also $\frac{3}{4}\div 1\frac{4}{7}=\frac{3}{4}\div\frac{11}{7}=\frac{3}{4}\cdot\frac{7}{11}=\frac{21}{44}$.

Similarly, $\dfrac{a}{b}\div\dfrac{c}{d}=\dfrac{a}{b}\cdot\dfrac{d}{c}=\dfrac{ad}{bc}$; and $\dfrac{a}{b}\div n=\dfrac{a}{b}\cdot\dfrac{1}{n}=\dfrac{a}{bn}$.

Also $\dfrac{a}{b}\div\left(c+\dfrac{n}{d}\right)=\dfrac{a}{b}\div\left(\dfrac{cd+n}{d}\right)=\dfrac{a}{b}\cdot\dfrac{d}{cd+n}=\dfrac{ad}{bcd+bn}$.

For division of fractions we have the

Rule. *Reduce all integral or mixed expressions to fractional form.*

Then invert the divisor, or divisors, and proceed as in multiplication of fractions.

EXERCISES

Perform the indicated operations:

1. $\frac{2}{3} \div \frac{5}{3}$.

2. $\frac{4}{5} \div \frac{9}{5}$.

3. $\frac{10}{3} \div \frac{5}{2}$.

4. $\frac{12}{15} \div (-\frac{4}{5})$.

5. $\frac{15}{21} \div -\frac{3}{7}$.

6. $\frac{18}{20} \div (-\frac{45}{10})$.

7. $\frac{a}{2x} \div \frac{c}{x}$.

8. $\frac{2a}{b} \div \left(\frac{-3c}{2b}\right)$.

9. $\frac{x^2}{y} \div \frac{2x}{y^2}$.

10. $\frac{2m}{n} \div \frac{-m}{3n}$.

11. $\frac{10x^2}{14y^2} \div \frac{15x^3}{21y^4}$.

12. $\frac{12ab^2}{5c} \div \frac{18a^2b}{15c^3}$.

13. $\frac{4a^2x}{51c} \div \frac{2ax}{17c^3}$.

14. $\left(\frac{3ab}{5c}\right)^2 \div \frac{18a^3b}{100c^3}$.

15. $\frac{a}{b} \div \frac{c}{b} \div \frac{a^2}{c^2}$.

16. $\frac{2x^6}{5a} \div \frac{4x^2}{15} \div \frac{(3x^2)^2}{12a^2}$.

17. $\frac{12c^3d}{35e^4} \div \frac{8c^2}{7e^2d} \cdot \frac{10de}{3c^2d^3}$.

Hint. See Rule, page 13.

18. $\frac{18x^3}{(7ab^2)^2} \div 12x^2 \cdot \frac{(14ab^4)^2}{6x^2}$.

19. $\left(\frac{3}{5x^2}\right)^2 \div \frac{256}{(10x^2)^2} \div \frac{5x^3}{(8x^4)^2}$.

20. $\frac{2a}{3b} \div \frac{(2a)^3}{12b^3} \div \frac{(7a^4)^2}{15b^5} \cdot \left(\frac{7}{5}\right)^2$.

21. $\frac{10-4a}{12x^2} \div \frac{5a-2a^2}{9x}$.

22. $\frac{(a-3)^2}{a^2-4a+3} \div \frac{a^2-9}{a^2-a}$.

23. $\frac{x^2+x-30}{x^2+5x-6} \div \frac{x^2-25}{x^2-6x+5}$.

24. $\frac{a^3-1}{a^3-a} \div \frac{a^4+a^3+a^2}{a^2+3a+2}$.

25. $(2m^2-4m+8) \div \frac{m^4+8m}{3m}$.

26. $\left(x+1-\frac{3}{4x}\right)\left(3+\frac{3}{4x^2-1}\right) \div \frac{2x+3}{2x+1}$.

27. $\left(\frac{-c}{2d^2}\right)^2 \div \left(\frac{-d}{c^3}\right)^2 \cdot \left(\frac{-d^3}{c^2}\right)^3 \div \left(\frac{d}{-2c}\right)^2$.

28. $\left(\frac{x^2}{y}-\frac{y^2}{x}\right)\div\left(\frac{x^2+xy+y^2}{2x^2-2xy}\right)\cdot\frac{1}{2(x-y)^2}.$

29. $\frac{x^2-5xy+6y^2}{9x^2-6xy+y^2}\cdot\left(3x+5y+\frac{10y^2}{x-2y}\right)\div\frac{x^2-9y^2}{3x^2+8xy-3y^2}.$

30. $\frac{(2m)^2}{m^6+64}\cdot\left(m^2-4+\frac{16}{m^2}\right)\div\frac{1}{2m}\div\frac{4m}{4-m^2}.$

31. $\frac{3x-2}{x+1}\cdot\left(3x+4+\frac{4}{3x}\right)\div\frac{9x^2-4}{9x^2+9x}.$

32. $\left(2x+1+\frac{1}{2x}\right)\div\frac{(8x^3-1)^2}{2x^2+2x}\cdot\left(4x-2+\frac{3}{x+1}\right).$

33. $\frac{27a^3+8b^3}{ax}\cdot\left(\frac{3a}{2b}\div\frac{2b}{3a}\right)\div\left[\left(3a-2b+\frac{4b^2}{3a}\right)\frac{9ab}{x}\right].$

81. Complex fractions. A complex fraction is a fractional expression containing one or more fractions either in the numerator, or in the denominator, or in both.

EXAMPLE

Simplify $\dfrac{1+\dfrac{8}{a}-\dfrac{9}{a^2}}{a-3+\dfrac{2}{a}}.$

Solution. Reducing the numerator and denominator to simple fractions,

$$\frac{1+\dfrac{8}{a}-\dfrac{9}{a^2}}{a-3+\dfrac{2}{a}}=\frac{\dfrac{a^2+8a-9}{a^2}}{\dfrac{a^2-3a+2}{a}}.$$

Performing the indicated division, the right member becomes

$$\frac{\cancel{(a-1)}(a+9)}{\underset{a}{\cancel{a^2}}}\cdot\frac{\cancel{a}}{(a-2)\cancel{(a-1)}}=\frac{a+9}{a^2-2a}.$$

Check. Letting $a = 3$,

$$\frac{1+\frac{8}{a}-\frac{9}{a^2}}{a-3+\frac{2}{a}} = \frac{a+9}{a^2-2a} \quad \text{becomes} \quad \frac{1+\frac{8}{3}-\frac{9}{9}}{3-3+\frac{2}{3}} = \frac{3+9}{9-6}, \text{ or } 4 = 4.$$

To simplify a complex fraction we have the

Rule. *Reduce both the numerator and the denominator to simple fractions, then perform the indicated division.*

EXERCISES

Simplify:

1. $\dfrac{3+\frac{1}{7}}{1-\frac{1}{7}}$.

2. $\dfrac{\frac{6}{7}-2}{1-\frac{3}{7}}$.

3. $\dfrac{9-\frac{4}{25}}{3-\frac{2}{5}}$.

4. $\dfrac{(\frac{3}{2})^2-16}{\frac{3}{2}+4}$.

5. $\dfrac{3^3-(\frac{1}{2})^3}{3^2+3\cdot\frac{1}{2}+(\frac{1}{2})^2}$.

6. $\dfrac{(\frac{3}{5})^2+\frac{3}{5}-2}{\frac{3}{5}-1}$.

7. $\dfrac{(\frac{2}{3})^3-(\frac{3}{2})^3}{(\frac{2}{3})^2+(\frac{2}{3})(\frac{3}{2})+(\frac{3}{2})^2}$.

8. $\dfrac{(\frac{7}{2})^3-(\frac{1}{3})^3}{(\frac{7}{2})^2+\frac{7}{2}\cdot\frac{1}{3}+(\frac{1}{3})^2}$.

9. $\dfrac{\frac{2}{3}}{5}+\dfrac{2}{\frac{3}{5}}$.

10. $\dfrac{c-\frac{1}{c}}{\frac{c-1}{c^2}}$.

11. $\dfrac{2-\frac{c^2}{2}}{\frac{2-c}{4}}$.

12. $\dfrac{x+\frac{x^2}{y}}{1+\frac{x}{y}}$.

13. $\dfrac{\frac{b^2}{5}-5}{1+\frac{b}{5}}$.

14. $\dfrac{\frac{c^2}{d^2}-\frac{d^2}{c^2}}{\frac{1}{c^2}+\frac{1}{d^2}}$.

15. $\dfrac{\frac{a^2}{b}-\frac{b^2}{a}}{a+b+\frac{b^2}{a}}$.

16. $\dfrac{\frac{x}{y}}{z}-\dfrac{x}{\frac{y}{z}}$.

17. $\dfrac{\frac{x}{y}+5+\frac{6y}{x}}{3+\frac{2x}{y}-\frac{9y}{x}}$.

18. $$\frac{\frac{8}{a} - a^2}{1 + \frac{2}{a} + \frac{a}{2}}.$$

19. $$\frac{1 - \frac{y}{x}}{\frac{x^2 + y^2}{x} - 2y}.$$

20. $$\frac{\frac{(x + 3y)^2}{6xy} - 2}{\left(\frac{1}{y} - \frac{3}{x}\right)^2}.$$

21. $$\frac{\left(\frac{3a - b}{2a^3}\right)^2}{\frac{(b - 3a)^2}{4a^2}}.$$

22. $$\frac{\frac{x - 2y}{2y} + \frac{x + 2y}{x}}{\frac{x - 2y}{x} - \frac{x + 2y}{2y}}.$$

23. $$\frac{\frac{1}{a} - \frac{b}{a + b}}{\frac{1}{a} + \frac{1}{b}}.$$

24. $$\frac{\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2}}{\frac{1}{x^3} + \frac{1}{y^3}}.$$

25. $$\frac{\frac{1}{5b^2} + \frac{1}{2ab} + \frac{1}{5a^2}}{\frac{1}{2a} + \frac{1}{b}}.$$

26. $$\frac{\frac{x}{x - 1} - \frac{2x}{x - 2}}{\frac{2x}{x - 2} - \frac{3x}{x - 3}}.$$

27. $$\frac{\frac{2(b + d) - 3(2a - c)}{2ab - bc + 2ad - cd}}{\frac{2}{2a - c} - \frac{3}{b + d}}.$$

28. $$\frac{\left(\frac{5x + 4y}{4x}\right)^2 - \frac{5y}{x}}{\frac{(5x + 4y)^2}{4x} - 20y}.$$

29. $$\frac{2 + \frac{12 - 6a}{2a^2 - a - 6}}{2a + \frac{21}{2a + 3} - 7}.$$

CHAPTER XVI

EQUATIONS CONTAINING FRACTIONS

82. Equations containing fractions with monomial denominators. If fractions are involved in one or both members of an equation, it is necessary to find a number or a literal expression by which one may multiply both members in order to get rid of the fractions. (Compare Problem 24, p. 145.) This process involves the application of Axiom III, page 40, which is the only principle employed in this chapter that has not been used repeatedly in the earlier work with equations.

Especial care is required to avoid error when a fraction which has two or more terms in its numerator is preceded by a minus sign.

ORAL EXERCISES

Solve for x, stating in each case the operations employed:

1. $\frac{x}{2} = 1.$

2. $\frac{x}{3} = 2.$

3. $\frac{x}{4} = 1.$

4. $\frac{x}{2} = 6.$

5. $\frac{3x}{2} = 4.$

6. $\frac{2x}{3} = 3.$

7. $\frac{2x}{5} = 10.$

8. $\frac{3x}{4} = 9.$

9. $\frac{x}{2} - 5 = 0.$

10. $\frac{x}{4} - 7 = 0.$

11. $\frac{x}{3} + 2 = 0.$

12. $\frac{x}{5} + 1 = 0.$

13. $\frac{2x}{3} - 8 = 0.$

14. $\frac{3x}{4} + 3 = 0.$

15. $\frac{5x}{3} - 10 = 0.$

16. $\frac{x+1}{2} = 2.$

17. $\frac{x-4}{5}=3.$

18. $\frac{x+6}{2}=8.$

19. $\frac{3x+1}{2}=2.$

20. $\frac{2x-1}{3}=5.$

21. $\frac{2x-5}{5}=3.$

22. $1=\frac{2}{x}.$

23. $3=\frac{6}{x}.$

24. $\frac{4}{x}=2.$

25. $\frac{10}{2x}=5.$

26. $\frac{6}{4x}=3.$

27. $\frac{5}{2x}=10.$

28. $\frac{4}{3x}=2.$

29. Using the least multiples possible, free Exercises **1–10**, page 177, of fractions.

EXAMPLES

1. Solve the equation $\frac{x}{3}+\frac{x}{5}=16.$

Solution. Multiplying both members by 15, the L.C.M. of the denominators, and canceling, we obtain

$$5x+3x=240,$$

or

$$8x=240.$$

$$x=30.$$

Check. Substituting 30 for x in the given equation,

$$\frac{30}{3}+\frac{30}{5}=16,$$

or

$$10+6=16.$$

2. Solve the equation $\frac{3}{4}(x+1)-\frac{5x-7}{6}=\frac{7}{3}.$

Solution. Removing the parenthesis,

$$\frac{3x}{4}+\frac{3}{4}-\frac{5x-7}{6}=\frac{7}{3}.$$

Multiplying both members by 12, the L.C.M. of the denominators, and canceling, with particular attention to the signs in the third fraction, we obtain

$$9x+9-10x+14=28.$$

Solving, $$x=-5.$$

Check. Substituting -5 for x in the given equation,

$$\frac{3}{4}(-5+1)-\frac{-25-7}{6}=\frac{7}{3}.$$

$$-3+\frac{16}{3}=\frac{7}{3}.$$

$$\frac{-9+16}{3}=\frac{7}{3}.$$

$$\frac{7}{3}=\frac{7}{3}.$$

For solving equations containing fractions with monomial denominators, we have the

Rule. *Free the equation of any parentheses it may contain.*

Find the L.C.M. of the denominators of the fractions and multiply each term of the equation by it, using cancellation wherever possible.

Transpose and solve as usual.

EXERCISES

Solve for x, and check results as directed by the teacher:

1. $\frac{x}{2}+\frac{x}{3}=10.$

2. $\frac{x}{4}+\frac{x}{8}=3.$

3. $\frac{x}{3}-\frac{x}{8}=5.$

4. $x+\frac{x}{5}=6.$

5. $\frac{x}{5}+\frac{x}{3}-\frac{x}{15}=14.$

6. $\frac{x+5}{4}-\frac{2x+4}{9}=1.$

7. $\frac{x-2}{2}+\frac{3x+2}{2}=6.$

8. $\frac{x+3}{4}+\frac{4x-5}{5}=5.$

9. $\frac{x+1}{2}+\frac{x-2}{3}-\frac{x+3}{4}=2.$

10. $\frac{2}{3}x+\frac{5}{4}x=\frac{23}{24}.$

11. $\frac{4}{3}x+\frac{2}{5}x=26.$

12. $\frac{3}{2}x+\frac{4}{3}x=\frac{17}{9}.$

13. $\frac{1}{x}+\frac{2}{x}=3.$

14. $\frac{3}{x}-\frac{2}{x}=6.$

15. $\frac{5}{x} + \frac{4}{x} = 9.$

16. $\frac{2}{x} + \frac{16}{15} = \frac{19}{3x}.$

17. $\frac{2x+3}{5} - \frac{1}{3}(x-3) = 2.$

18. $\frac{5-x}{2} + \frac{2x+3}{3} = 4.$

19. $\frac{2x+13}{3} - \frac{6-x}{4} = 1.$

20. $\frac{3}{4}(x-1) + \frac{5x-7}{4} = \frac{3}{2}.$

21. $\frac{5x+12}{6} - \frac{4}{11}(2x+7) + \frac{1}{3} = 0.$

22. $\frac{x-7}{6} + \frac{1}{2}\left(2 - \frac{x}{2}\right) = \frac{3x-22}{6}.$

23. $\frac{x}{a} + x = a + 1.$

24. $\frac{x}{a} - \frac{x}{3a} = 6.$

25. $\frac{x}{a} - \frac{2x}{5a} = \frac{3}{5b}.$

26. $6 - \frac{x}{c} = \frac{1}{3}(c - 3x) + \frac{17c}{3}.$

27. $n\left(\frac{2}{3} - \frac{n}{5}\right) - \frac{x}{3} = \frac{n}{5}(5n - 3x).$

28. $2x + 2a + \frac{3x+4}{a} = \frac{2(2a^2-1)}{a} - 1.$

29. $\frac{x+a}{a} - \frac{x+b}{b} = a^2(b-a).$

30. $\frac{2a-3x}{6a} + \frac{5a-2x}{5a} + \frac{41}{30} = 0.$

31. $\frac{6x}{a} - \frac{1}{2}\left(\frac{2a}{3} - 12x\right) + \frac{19a}{3} = -6.$

32. $(x+5)(x-6) = x\left(x - \frac{5}{2}\right).$

33. $(x+7)(x+18) = x\left(x - \frac{1}{5}\right).$

34. $(4x+3)(6+x) = 4x\left(x - \frac{1}{6}\right) + 45\frac{2}{3}.$

PROBLEMS

1. One third of a certain number, plus $\frac{1}{12}$ of that number, equals 15. Find the number.

HINT. $\frac{n}{3}+\frac{n}{12}=15$.

2. The difference between $\frac{1}{4}$ of a certain number and $\frac{1}{9}$ of that number is 10. Find the number.

3. The sum of two numbers is 38. One tenth of the greater number equals $\frac{1}{9}$ of the less. Find the numbers.

4. The width of a rectangle is $\frac{2}{3}$ of its length. The perimeter is 200 meters. Find the area of the rectangle.

5. What number must be added to the numerator of the fraction $\frac{4}{7}$ so that the resulting fraction will be $\frac{1}{5}$ of the number in question?

6. One half of a certain integer is $\frac{1}{5}$ the sum of the next two consecutive integers. Find the first integer.

7. A certain even integer divided by 12 is equal to $\frac{1}{26}$ of the sum of the next two consecutive even integers. Find the first integer.

8. What number added to both terms of the fraction $\frac{5}{7}$ gives a fraction whose value is $\frac{5}{6}$?

9. Separate 24 into two parts such that $\frac{1}{4}$ of their difference is 4.

10. One fourth the difference of three times a certain number and 4 equals $\frac{1}{7}$ the difference of five times the number and 4. Find the number.

11. Separate 108 into two parts such that their quotient is $\frac{2}{7}$.

12. There are two numbers whose sum is 24. If their difference be divided by their sum, the quotient will be $\frac{1}{60}$ of the greater number. Find the numbers.

13. A's age is $\frac{5}{2}$ B's age. In 10 years A's age will be twice B's age. Find their ages now.

14. At the time of marriage a certain woman's age was $\frac{4}{5}$ that of her husband. Twenty years later her age was $\frac{8}{9}$ of his. Find their ages at the time of marriage.

15. A is 12 years older than B. Eight years ago B was $\frac{5}{8}$ as old as A. Find their ages now.

16. Jupiter has 5 more moons than Uranus, and Saturn two more than twice as many as Uranus; Mars has 7 fewer than Jupiter, and Neptune half as many as Mars. These planets together have 26 moons. How many has each?

17. A triangle has the same area as a trapezoid. The altitude of the triangle is 24 feet and its base is 12 feet. The altitude of the trapezoid is a third that of the triangle, and one of its bases equals the base of the triangle. Find the other base of the trapezoid.

In solving Problems 18 and 19, the student should make a table similar to those shown on pages 100 and 101.

18. A marksman hears the bullet strike the target 3 seconds after the report of his rifle. If the average velocity of the bullet is 1925 feet per second and the velocity of sound is 1100 feet per second, find the distance to the target and the length of time the bullet was in the air.

19. A gunner using one of the best modern rifles would hear the projectile strike a target 2640 yards distant $9\frac{2}{5}$ seconds after the report of the gun, provided the projectile maintained throughout its flight the same velocity it had on leaving the gun. Find this velocity if sound travels 1100 feet per second.

83. Equations containing fractions with polynomial denominators. Although no new principle is involved in the exercises of this section, they serve to review many of the most important processes of algebra.

ORAL EXERCISES

Clear the following equations of fractions, stating in each case the operation employed. Do not solve the equations.

1. $\frac{x}{x+1} = 3.$

Solution. Multiplying each member of the equation by $x + 1$ we obtain the equation $x = 3x + 3$.

2. $\frac{x}{x-1} = 4.$
3. $\frac{2x}{1-x} = \frac{5}{2}.$
4. $\frac{3x}{x^2+4} = 1.$
5. $\frac{x-1}{x-2} = 3.$
6. $\frac{x+1}{x-5} = 6.$
7. $\frac{x+1}{x-1} = \frac{4}{3}.$
8. $\frac{x+7}{x-3} = \frac{5x}{3}.$
9. $\frac{x-3}{x+2} = \frac{3}{4x}.$
10. $\frac{x+1}{x+2} = \frac{x+2}{x-1}.$
11. $\frac{x+2}{x+3} = \frac{x-2}{x-1}.$
12. $\frac{x-4}{x+3} = \frac{x-5}{x+2}.$
13. $\frac{1}{x} = x - 1.$
14. $\frac{1}{x} = \frac{2}{x-1}.$
15. $\frac{x-1}{x} = \frac{x}{4}.$
16. $\frac{x}{x-1} = \frac{5}{x}.$
17. $\frac{x-2}{3x} = \frac{1}{x}.$
18. $\frac{4x}{1-2x} = \frac{2x}{1+x}.$
19. $\frac{3-x}{3x-1} = \frac{x}{2-x}.$

EXERCISES

Solve the following equations and check:

1. $2 + \frac{x-1}{x+3} = \frac{11}{5}.$

Solution. Multiply both members of the equation by $5(x + 3)$, the L.C.M. of the denominators. This may be indicated symbolically as follows:

$$\left[2 + \frac{x-1}{x+3} = \frac{11}{5}\right] 5(x+3).$$

By means of this form of expression one can readily see what ought to be canceled.

Canceling, $10(x + 3) + 5(x - 1) = 11(x + 3)$,

or $10x + 30 + 5x - 5 = 11x + 33.$

Transposing and collecting, $4x = 8$,

whence $x = 2.$

Check.

$$2+\frac{2-1}{2+3}=\frac{11}{5}.$$

$$2+\tfrac{1}{5}=\tfrac{11}{5}.$$

$$\tfrac{11}{5}=\tfrac{11}{5}.$$

2. $\dfrac{x-3}{3+x}=4.$ **3.** $\dfrac{x-5}{x-2}=\dfrac{1}{2}.$ **4.** $\dfrac{3x+4}{2x-2}=-2.$

5. $\dfrac{4}{6-5x}=\dfrac{1}{x}.$ **6.** $\dfrac{5x}{3}=\dfrac{3-x}{2+x}+x.$

Solution. The L.C.M. of the denominators is $3(2+x)$. Hence from an inspection of

$$\left[\frac{5x}{3}=\frac{3-x}{2+x}+x\right]3(2+x)$$

we obtain $5x(2+x)=3(3-x)+3x(2+x),$

or $10x+5x^2=9-3x+6x+3x^2.$

Transposing and collecting,

$$2x^2+7x-9=0.$$

Factoring, $(2x+9)(x-1)=0,$

whence $x=-\tfrac{9}{2}, 1.$

Check as usual.

7. $\dfrac{x-3}{x}=\dfrac{2x}{x-2}.$

8. $\dfrac{x-1}{x+3}=x.$

9. $\dfrac{x}{5-x}=\dfrac{2x}{4x-5}.$

10. $\dfrac{2}{x+1}=\dfrac{4}{4+x}.$

11. $\dfrac{1}{2-x}=\dfrac{3}{3-x}.$

12. $\dfrac{x-5}{4-5x}=-6x.$

13. $\dfrac{x+7}{3-x}+\dfrac{3x}{2}=x.$

14. $\dfrac{2}{x-6}=\dfrac{x-12}{x+6}+\dfrac{x}{x+6}.$

15. $\dfrac{x+3}{x-4}=\dfrac{x+9}{x-5}.$

16. $\dfrac{4x-1}{x+2}=\dfrac{7-2x}{x+4}.$

17. $\dfrac{4}{2x-3}+\dfrac{3x+2}{3}=x.$

18. $\dfrac{x+6}{3-x}=\dfrac{5+2x}{7x-5}.$

19. $\dfrac{3x}{8}-\dfrac{4}{x-4}=\dfrac{3x-4}{8}.$

20. $\dfrac{2}{4x+1}-\dfrac{1}{5x-2}-\dfrac{1}{3x+4}=0.$

21. $\frac{47}{220} - \frac{7}{4(x+3)} = \frac{3}{5(x+3)}$.

22. $\frac{4x}{x+3} - \frac{6}{2(x+3)} = \frac{x^2+11}{3(x+3)}$.

23. $\frac{4-x}{x-5} + \frac{7}{5} = \frac{3}{5-x}$.

HINT. Multiply the terms of the last fraction by -1.

24. $\frac{x}{x-3} + 3 = \frac{1}{3-x}$.

25. $\frac{3x}{x-2} = \frac{x-3}{2-x} - x$.

26. $\frac{3x}{x-3} - \frac{16}{x^2-9} = \frac{x}{x+3} + \frac{2x}{9-x^2}$.

HINT. Rewriting the last fraction (see Hint after Exercise 23), and factoring the denominators,

$$\frac{3x}{x-3} - \frac{16}{(x-3)(x+3)} = \frac{x}{x+3} - \frac{2x}{(x-3)(x+3)}.$$

The L.C.M. of the denominators is $(x-3)(x+3)$. Hence from

$$\left[\frac{3x}{x-3} - \frac{16}{(x-3)(x+3)} = \frac{x}{x+3} - \frac{2x}{(x-3)(x+3)}\right](x-3)(x+3)$$

we obtain $$3x(x+3) - 16 = x(x-3) - 2x,$$

which may be solved and checked as usual.

In solving equations containing fractions with polynomial denominators the student should write the denominators and their L.C.M. in factored form, as in the preceding work. With this exception the rule on page 177 applies to all equations containing fractions.

27. $\frac{1}{3-x} + \frac{2}{x+3} = \frac{6x}{x^2-9}$.

28. $\frac{2x}{x+1} + \frac{3}{x-1} + \frac{24}{1-x^2} = 0$.

29. $\frac{3}{x-4} = \frac{5x-12}{x^2-16} - \frac{4}{16-x^2}$.

30. $\frac{x+2}{x-2}=\frac{10-2x^2}{4-x^2}-\frac{7}{x^2-4}.$

31. $\frac{1+x}{3-x}+\frac{x-2}{2-x}=\frac{x-1}{x^2-5x+6}.$

32. $\frac{x-4}{2x-5}+\frac{2x-15}{2x+4}=\frac{8x^2-20x-31}{4x^2-2x-20}.$

84. Equations containing decimals. The method of solving an equation containing decimals is illustrated in the following:

EXAMPLE

Solve the equation $.3x+.7=4.2-.05x$.

Solution. Multiplying each member of the equation by 100,

$$30x+70=420-5x.$$

Transposing and combining,

$$35x=350.$$

$$x=10.$$

Check.

$$.3\times 10+.7=4.2-.05\times 10.$$

$$3+.7=4.2-.5.$$

$$3.7=3.7.$$

In equations containing fractions, if decimals occur in any denominator, multiply both numerator and denominator of such a fraction by that power of ten which will reduce the decimals in both the numerator and the denominator to integers. Then clear of decimals and proceed as in the foregoing example. (See Hint after Exercise 20, p. 185.)

EXERCISES

Solve and check the following equations:

1. $.4x=6.$
2. $.3x+.5=.8.$
3. $.3x+4=.25.$
4. $.75-.7x=.26.$
5. $.92+.3x=5.12.$
6. $3.75=2.15-.5x.$

7. $.06x - 4.5 = 1.68.$

8. $.03x + .16 = .58.$

9. $.04x = .1x + 2.4.$

10. $.8x - 2.7 = .55x.$

11. $.15x - .4x = 8x - 49.5.$

12. $1.7x + 3.14 = -9.66 - 1.5x.$

13. $3x + 7 - 1.25x = 8.845 + .52x.$

14. $1.7x + .17 - .03x = 4.73 + 1.1x.$

15. $.12(2x + .5) - .2(1.5x - 2) = .4.$

16. $6(3x - 1.1) - 8.4(.7x - 3) = 6x + .24.$

17. $\frac{.7x}{4} - \frac{.19}{3} = \frac{.1x}{6}.$

18. $\frac{.3x - 6.2}{4} - \frac{6.75 - .4x}{5} = \frac{3.5}{2}.$

19. $\frac{.14x + 3.2}{6} - \frac{x - .75}{5} = \frac{2.4x - 1.5}{10}.$

20. $\frac{6x - 2.69}{.4} + \frac{1.5x}{.24} = 3.9.$

Hint. Multiplying numerator and denominator of each fraction by 100,

$$\frac{600x - 269}{40} + \frac{150x}{24} = 3.9.$$

Solve and check as usual.

To avoid the possibility of repeating, in the check, a numerical error made in the solution, the check should be performed by finding the value of each fraction separately, without clearing of fractions.

21. $\frac{1.8x - 2}{1.7} = \frac{5.7 - .7x}{1.8}.$

22. $\frac{3.2x}{.5} + \frac{4.5x}{12.5} = 1.52 + 6x.$

23. $\frac{.3(3 - .4x)}{.16} - \frac{.5(.2x - 6)}{.8} = 5.$

24. $\frac{.3(5 - x)}{6.25} = \frac{1.5 - 10x}{14} + .56x.$

NOTE. The introduction into Europe of the Arabic notation for numbers was one of the important events of the Middle Ages. This notation originated among the Hindus at least as early as A.D. 700. It was adopted by the Arabs, and was introduced by the Moors into Spain during the twelfth and thirteenth centuries. Anyone who has tried to multiply two numbers in the Roman notation, like MDCCVII by MCXVIII, will realize the difficulties that surrounded arithmetical operations before the Arabic system was taught. Before the introduction of this system one of the principal uses for arithmetic was the determination of the day of the month on which Easter came. Roger Bacon in the thirteenth century urged the theologians "to abound in the power of numbering," so that they might carry out these computations. Business computations were made on the abacus, one form of which was a contrivance of wires and sliding balls on which arithmetical operations can be performed with great rapidity.

Though computation in the decimal system was common in Europe from the thirteenth century, the final step in perfecting the notation was not taken until about 1600, when Sir John Napier and others made use of the decimal point in the modern sense. It was not until the beginning of the eighteenth century that it came into general use.

85. Percentage. The methods of algebra may be used to advantage in treating many problems in percentage which are also found in arithmetic, and in solving many others which would be difficult or impossible to solve by arithmetical means alone.

ORAL EXERCISES

1. What is 4% of 60?

2. What is 4% of x?

3. What is 3% of $x + 40$?

4. What is 6% of $4x - a$?

5. What is the interest on $100 left for one year at 5%?

6. What is the interest on $100 left for 6 years at 5%?

7. What is the interest on $100 left for t years at 6%?

8. What is the interest on P dollars left for t years at 6%?

9. What is the interest on P dollars left for t years at $r\%$?

10. What is the total amount due at the end of t years if P dollars are left at $r\%$?

If two sums of money are x dollars and $2500 - x$ dollars respectively, express as an equation the statement made in each of Exercises 11–14.

11. Four per cent of the first sum equals $24.

12. Five per cent of the first sum plus 6% of the second sum equals $130.

13. Six per cent of the first sum equals 4% of the second.

14. Five per cent of the first sum is $44 more than 4% of the second.

When P dollars are left at simple interest for t years, at a yearly rate of $r\%$, the total amount A accumulated is given by the following formula:

$$P+(P \cdot r \cdot t)=P(1+rt)=A.$$

It should be noted carefully that the value of r is a fraction of which the denominator is 100. Thus, if the rate is 5%, the value of r is $\frac{5}{100}$, or .05.

EXAMPLE

What sum of money placed at simple interest for 2 years at 5% amounts to $99?

Solution. Principal + interest = 99.

Let $P =$ the principal.

Then $.05P \cdot 2 =$ the interest for two years.

Therefore $P + .10P = 99$,

or $1.10P = 99.$

Whence $P = 90.$

Check. $90 + 9 = 99.$

EXERCISES

1. What sum of money placed at interest for 1 year at 6% amounts to \$371?

2. What sum of money placed at simple interest for 3 years at 4% will amount to \$476?

3. In how many years will \$325, at 6% simple interest, gain \$39?

4. In how many years will \$480, at $6\frac{1}{2}$% simple interest, gain \$156?

5. At what per cent simple interest will \$375 gain \$75 in 4 years?

Solution. 375 × the rate of interest × 4 = 75.

Let $x =$ the rate of interest.

Then \$$375x =$ the interest for 1 year,

and \$$1500x =$ the interest for 4 years.

Therefore $1500x = 75$,

and $x = \frac{1}{20}$, or $\frac{5}{100}$.

Hence the money is lent at 5%.

6. At what per cent simple interest will \$725 gain \$145 in 4 years?

7. At what per cent simple interest will \$250 amount to \$317.50 in 6 years?

8. In how many years will \$300 double itself at 5% simple interest?

9. In how many years will \$500 treble itself at 6% simple interest?

10. A part of \$800 is invested at 3%, and the remainder at 4%. The yearly income from the two investments is \$30. Find each investment.

Solution. One part × .03 + the other part × .04 = 30.

Let $x =$ the number of dollars invested at 3%.

Then $800 - x =$ the number of dollars invested at 4%.

Therefore, by the conditions of the problem,

$$.03\,x + .04\,(800 - x) = 30.$$

Multiplying each member by 100,

$$3\,x + 4\,(800 - x) = 3000.$$

Solving, $x = 200,$

and $800 - x = 600.$

Hence the 3% investment is \$200, and the 4% investment is \$600.

Check.

$$\begin{array}{rr} 200 & 600 \\ \underline{.03} & \underline{.04} \\ 6.00 & 24.00 \end{array} \qquad \$6 + \$24 = \$30.$$

11. A part of \$1500 is invested at 5% and the remainder at 4%. The total annual income from the two investments is \$69. Find the amount of each investment.

12. A sum of money at 6% interest and a second sum at 5% yield a total annual income of \$50. The first sum exceeds the second by \$100. Find each.

13. A 4% investment yields annually just as much as one at 5%. If the sum of the investments is \$3600, find each.

14. A 5% investment yields annually \$10 less than a 6% investment. If the sum of the two investments is \$240, find each.

15. A man invests part of \$4300 at 6% and the remainder at 5%. The investment at 6% yields annually \$1.60 less than the one at 5%. Find the sum invested at 5%.

16. A man invests part of \$5360 at 5% and the remainder at 6%. The yearly income from the 5% investment is \$63.40 more than that from the 6% investment. Find the sum invested at 6%.

17. A part of \$4560 is invested at 4% and the remainder at 6%. The total yearly income is \$202.40. Find the amount invested at 6%.

86. Literal equations. At this point the student should review the solutions on pages 95 and 96.

EXERCISES

Solve for x, and check as directed by the teacher:

1. $2ax - 2a^2 = 6a^2 - 2ax.$
2. $5c^2 - 4cx = c(5x - 4c).$
3. $3a^2x - 4b^2 = b^2 - 2a^2x.$
4. $2(x + 3) - 4a = 6.$
5. $3(x - 1) - 9a = 6.$
6. $3(x - 2) - 4h = 2(x - 2).$
7. $3(2x - a) = 2(x - a).$
8. $4(x - b) = 2(x + b).$
9. $4a(6x - 3b) = 3b(8 - 4a).$
10. $ax + bx = a^2 + ab.$
11. $cx + b^2 = bx + bc.$
12. $ax + b^2 = a^2 - bx.$
13. $6ab + 15cx - 10bc - 9ax = 0.$
14. $12 - 15a + 16x = 20ax.$
15. $ax + bx + cx = ak + bk + ck.$
16. $2ax + 2a + cx + c + x + 1 = 0.$
17. $6ab + kx + 4a^2 = 2ax + 3bk + 2ak.$
18. $5ax - 5a^2 - 10ab = 3ac + 6bc - 3cx.$
19. $\frac{x}{2b} = a.$
20. $\frac{3ab}{x} = b.$
21. $\frac{c}{x} + \frac{3c}{2x} = \frac{5}{4}.$
22. $\frac{8a}{3x} + \frac{8a}{x} = \frac{3}{2} + \frac{5a}{3x}.$
23. $\frac{x}{a} + \frac{x}{b} = a + b.$
24. $\frac{d^2}{x} - d = \frac{a^2}{x} + a.$
25. $\frac{hx}{2k} - 4k^2 = \frac{2kx}{h} - h^2.$
26. $\frac{2x}{a} + \frac{a - 4x}{3} - 3a = -4.$
27. $\frac{x}{a} + \frac{x}{c} + ac = bc + ab + \frac{x}{b}.$
28. $\frac{3a - 4x}{5a} + \frac{3a - 2x}{4a} = \frac{1}{20}.$
29. $\frac{bx}{2} - \frac{3a}{5}\left(x - \frac{2ab}{3}\right) = ab\left(\frac{b}{2} - \frac{a}{5}\right).$
30. $\frac{x - a}{x - b} = \frac{b}{a}.$

31. $\dfrac{b}{b(b-x)}+\dfrac{3}{d(b-x)}+\dfrac{d+3}{2\,db}=0.$

32. $\dfrac{c}{k(x+c)}+\dfrac{k}{c(x-k)}=\dfrac{c^2-kc+2\,k^2}{2\,kc(x-k)}.$

33. $\dfrac{1}{ab}+1-\dfrac{ab}{x}=\dfrac{1}{x}.$

87. Meaning of primes and subscripts. Different but related values are often represented by the same letter, with smaller characters written at the right and above or below the letter used; as, y', y'', x_0, $4\,x_3$, t_m^2, t_w. These are read *y prime, y second, x sub zero, 4 x sub three, the square of t sub m, and t sub w* respectively. Primes and subscripts, unlike exponents, possess no numerical significance, and the student should carefully note that x_0 and x_3 are as different numerically as a and b.

The notation just explained is very convenient in physics, where L_1 and L_2 may denote different but related lengths; W_1 and W_2 may represent two different weights; and t_0, t_1, and t_2 may mean three unequal but related intervals of time.

Primes are cumbersome and easily confused with exponents; hence subscripts are preferable.

The following equations are taken from algebra, geometry, and physics, where it is often necessary to express one of the quantities (weight, time, distance, etc.) in terms of the others.

EXERCISES

1. Solve for R; $K=2\,\pi RH.$
(Formula for curved surface of cylinder.)

2. Solve for a; $A=\dfrac{ab}{2}.$
(Formula for area of triangle.)

3. Solve for R; $C=2\,\pi R.$
(Formula for circumference of circle. π = about 22/7.)

4. Solve for r and t; $d = rt$.
(Formula for uniform motion.)

5. Solve for a and A; $\frac{a}{A} = \frac{D}{360}$.
(Formula relating to the measurement of angles.)

6. Solve for C; $\frac{D}{360} = \frac{l}{C}$.

7. Solve for r; $C = \frac{E}{R + r}$.
(Ohm's law for a simple electrical circuit.)

8. Solve for r and n; $C = \frac{E}{R + nr}$.

9. Solve for r and n; $C = \frac{n \cdot e}{R + nr}$.

10. Solve for F; $C = \frac{5}{9}(F - 32)$.
(Formula for converting thermometer readings from one scale (Fahrenheit) to another (Centigrade).)

11. Solve for W_2; $\frac{W_1}{W_2} = \frac{L_2}{L_1}$.

12. Solve for r and t; $A = P(1 + rt)$.

13. Solve for P_2; $\frac{V_1}{V_2} = \frac{P_2}{P_1}$.
(Formula relating to volume and pressure of a gas.)

14. Solve for n and l; $s = \frac{n(a + l)}{2}$.

15. Solve for a, l, and r; $s = \frac{rl - a}{r - 1}$.

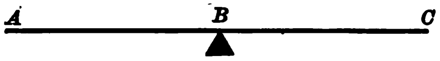
16. Solve for a; $\frac{D}{180} = \frac{a}{\pi}$.

17. Solve for t_1; $V_1 = V_0(1 + .00365\,t_1)$.

18. Solve for b_2; $A = \frac{(b_1 + b_2)a}{2}$.

88. The lever. The figure given below is a diagram of a machine called a *lever*. AC is a stiff bar resting on a single support at B. This support is called the *fulcrum* and AB and BC are spoken of as *arms* of the lever.

Those who have played with a teeter board have had some experience with a lever, and they have found that, in order to balance, the heavier of two persons must sit nearer the fulcrum than the lighter one does.



In general, if the lengths of the arms of a lever are l_1 and l_2 and the corresponding weights are W_1 and W_2, a balance results when

$$l_1W_1 = l_2W_2.$$

Thus, if $AB = 3$ feet and $BC = 4$ feet, a boy at A who weighs 100 pounds will balance a boy at C who weighs 75 pounds; for $3 \cdot 100 = 4 \cdot 75$.

PROBLEMS

1. A, who is 4 feet from the fulcrum, balances B, who is 6 feet from it. A weighs 96 pounds. Find the weight of B.

2. A, who weighs 100 pounds, balances B, who weighs 120 pounds. B is 80 inches from the fulcrum. How far from it is A?

3. A, who weighs 125 pounds, balances B, who weighs 100 pounds. The distance between them is 9 feet. How far is each from the fulcrum?

4. A and B together weigh 210 pounds. They balance when A is 3 feet 9 inches from the fulcrum and B is 5 feet from it. Find the weight of each.

5. A weighs $\frac{2}{3}$ as much as B. The distance between them is 24 feet. How far from B is the fulcrum if they balance?

REVIEW PROBLEMS

1. Separate 240 into two parts such that their quotient is 7.

2. Separate 68 into two parts such that $\frac{2}{3}$ of the greater shall equal $\frac{3}{4}$ of the less.

3. Separate $\frac{4}{3}$ into two parts such that $\frac{1}{3}$ of one part shall equal $\frac{1}{5}$ of the other.

4. Find two numbers whose sum is 98 and such that the greater divided by the less gives a partial quotient of 3 and a remainder of 6.

HINTS. $$\frac{\text{Dividend}}{\text{Divisor}} = \text{Partial Quotient} + \frac{\text{Remainder}}{\text{Divisor}}.$$

Let $x =$ the less number.

Then $98 - x =$ the greater number.

That is, $$\frac{98 - x}{x} = 3 + \frac{6}{x}.$$

Solve and check as usual.

5. Separate 120 into two parts such that one divided by the other gives a partial quotient of 5 and a remainder of 12.

6. The sum of two numbers is 1797. The greater divided by the less gives a partial quotient of 70 and a remainder of 22. Find the numbers.

7. Separate $\frac{11}{6}$ into two parts such that their product is greater by $\frac{1}{3}$ than the square of the smaller part.

8. The sum of two numbers is 16. Four times the greater number exceeds 50 by half as much as 28 exceeds the less. Find the numbers.

9. A boy's age now is $\frac{3}{5}$ of what it will be 10 years hence. How old is he now?

10. One sixth of a certain man's age 12 years ago equals $\frac{1}{8}$ of his age 8 years hence. What is his age now?

11. A collection of nickels and quarters contains 60 coins. Their total value is $11. How many are there of each?

12. Twenty-three coins, dimes and quarters, have the value $3.05. How many are there of each?

13. The square of half a certain even number is 14 less than $\frac{1}{4}$ the product of the next two consecutive even numbers. Find the numbers.

14. A rectangle is 4 times as long as it is wide. If it were 4 feet shorter and $1\frac{1}{2}$ feet wider, its area would be 11 square feet more. Find its length and breadth.

15. A rectangle is $\frac{3}{8}$ as broad as it is long. If its length were doubled and its breadth diminished by 14, its area would be diminished by the square of half its breadth. What are its dimensions?

16. It costs as much to sod a square piece of ground at 15 cents per square yard as to fence it at 20 cents per foot. Find the side of the square.

17. A rectangular court is twice as long as it is wide. It costs half as much to fence it at 50 cents per yard as to seed it at 15 cents per square yard. Find its dimensions.

18. A rectangular picture $2\frac{1}{2}$ times as long as wide is surrounded by a frame 2 inches wide. The area of the frame is 128 square inches. Find the dimensions of the picture.

19. A man bought apples at 16 cents per dozen. He sold $\frac{1}{6}$ of them at the rate of 3 for 5 cents, but the rest were not so good, and he had to sell them at the rate of 2 for 3 cents. He made a profit of 48 cents on the entire transaction. How many dozen apples did he buy?

20. A can do a piece of work in 2 days, B in 3 days, and C in 4 days. How long will it take them, working together?

Solution. By the conditions of the problem A does $\frac{1}{2}$ of the work in one day, B does $\frac{1}{3}$ of the work in one day, and C does $\frac{1}{4}$ of the work in one day. Let x represent the number of days required by A, B, and C together to do the work.

Then $\frac{1}{x}$ = the fractional part of the work the three together do in one day.

Therefore $$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{1}{x}.$$

Solving, $$x=1\tfrac{1}{13}.$$

Check. $$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{1}{1\frac{1}{13}}, \text{ or } \frac{13}{12}=\frac{13}{12}.$$

21. A can do a piece of work in 6 days, and B in 9 days. How many days will they require, working together?

22. A can do a piece of work in 6 days, B and C each in 8 days, D in 12 days. How many days will they require, working together?

23. A can do a piece of work in 3 days, B in $4\frac{1}{2}$ days. How many days will they require, working together?

24. A can do a piece of work in 8 days, A and B working together in $4\frac{4}{5}$ days. How long would it take B alone?

25. A can do a piece of work in $5\frac{1}{4}$ days, B in $4\frac{1}{5}$ days, A, B, and C together in $1\frac{3}{4}$ days. How long would it take C alone?

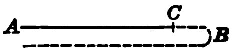
26. A can do a piece of work in 8 days. After he has worked 3 days, B joins him and they finish the work in 3 more days. How long would it have taken B to do the work alone?

Hint. What fractional part does A do in 1 day? in 3 days? in 6 days? What fractional part does B do in 1 day? in 3 days?

27. A can do a piece of work in 4 days, B in 5 days. After A has worked alone for 1 day, B joins him and they finish the job together. How much longer does it take them?

28. Two bicyclists start at the same time to ride from A to B, 80 miles distant. One travels 4 miles an hour more than the other. The faster bicyclist reaches B and at once returns, meeting the slower one at C, 64 miles from A. Find the rate of each.

Solution. The problem states that the two travel at different rates, that they travel different distances, but that the time is the same for each. Hence the equation must be formed by expressing the time t, or d/r, for each, and equating the two expressions for t.



The men together cover twice the distance from A to B, or 160 miles. As the slower one travels 64 miles, the faster travels 160 − 64, or 96, miles. If x equals the rate of the slower bicyclist in miles per hour, we have:

	d in miles	r in miles per hour	$\frac{d}{r} = t$ in hours
Slower bicyclist	64	x	$\frac{64}{x}$
Faster bicyclist	$160 - 64 = 96$	$x + 4$	$\frac{96}{x+4}$

Hence
$$\frac{64}{x} = \frac{96}{x+4}.$$

Solving, we obtain $x = 8$, the rate of the slower bicyclist in miles per hour, and $x + 4 = 12$, the rate of the faster bicyclist.

Check. $\frac{64}{8} = 8$, and $\frac{96}{12} = 8$.

29. Two bicyclists, A and B, start at the same time to ride from X to Y, 60 miles distant. A travels 4 miles per hour less than B. The latter reaches Y and at once turns back, meeting A 12 miles from Y. Find the rate of each.

30. A train runs 360 miles. On the return trip it increases its rate by 5 miles an hour and makes the run in an hour less time. Find the rates going and returning.

31. An automobile makes a run of 120 miles. The chauffeur then increases the speed by 5 miles an hour and returns over the same route in 4 hours less time. Find the rates going and returning.

32. A bicyclist traveling 15 miles per hour was overtaken $11\frac{1}{3}$ hours after he started by an automobile which left the same starting point 3 hours and 20 minutes later. What was the rate of the automobile?

33. A man travels at a uniform rate from A to B, 120 miles distant. He travels the first 70 miles without stopping. The remainder of the journey, including a delay of 2 hours, requires the same time as the first part. Find his rate.

Solution. By reading the problem we discover that the distances covered in the first and second portions of the journey are different, that the time of travel is not the same for each, but that the rate throughout is the same. Hence the equation will be formed by finding two expressions for the rate r, or d/t, and setting them equal to each other. If x equals the number of hours required to travel 70 miles, we have:

	d in miles	t in hours	$d/t = r$ in miles per hour
First part of journey	70	x	$\frac{70}{x}$
Second part of journey	50	$x - 2$	$\frac{50}{x-2}$

Hence $$\frac{70}{x} = \frac{50}{x-2}.$$

Solving, we obtain $x = 7$, the time in hours occupied in traveling the first 70 miles, and $70 \div 7 = 10$, the rate in miles per hour.

Check. $70 \div 10 = 7$, and $7 - 5 = 2$.

34. A leaves a certain point and walks at the rate of $3\frac{1}{2}$ miles per hour. Two and a half hours later B leaves the same point and drives in the opposite direction at the rate of $9\frac{1}{2}$ miles per hour. How much time must elapse after A starts before they will be 25 miles apart?

35. A and B start at the same time from two points 120 miles apart and travel toward each other. A's rate is 2 miles per hour less than B's. The latter, having been delayed $2\frac{1}{2}$ hours on the way, has traveled the same distance as A when they meet. Find the rate of each.

36. A man rows 4 miles per hour in still water. He finds that it requires 5 hours to row upstream a distance which he can row downstream in 3 hours. Find the rate of the current.

HINT. Let $x =$ the rate of the current. Then $4 - x =$ the rate upstream, and $4 + x =$ the rate downstream.

37. A man who can row $4\frac{1}{2}$ miles per hour in still water rows up a stream the rate of whose current is 2 miles per hour. After rowing back he finds that the entire trip took 6 hours. How far upstream did he go?

38. A man who can row 4 miles an hour in still water rows downstream and returns. The rate of the current is $1\frac{1}{3}$ miles per hour, and the time required for the round trip is 15 hours. How many hours did he take to return?

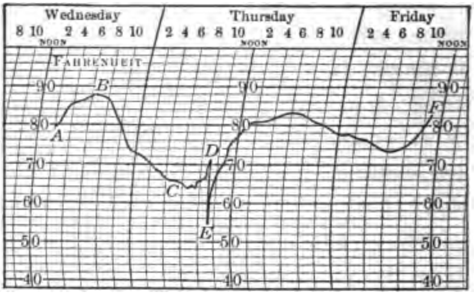
39. French scouts observe a German aëroplane leave its base, fly 24 miles due west over the French lines, and return to its starting point. They are prevented by an encounter with a German detachment from observing the time of either part of the trip, but they are able to observe that the duration of the round trip was 40 minutes. At the time there was an east wind blowing, with a velocity of 15 miles per hour. What was the velocity of the aëroplane?

40. A farmer pays \$48 for a flock of sheep. He sells all but two of them for \$50, and gains a dollar on each sheep sold. How many sheep had he?

CHAPTER XVII

GRAPHICAL REPRESENTATION

89. Temperature curve. The curve *ABCDEF* is called **a graph.** It was made by a recording thermometer. Such an instrument is provided with an arm carrying a pen, which moves up as the temperature rises, and down as it falls. A clock movement runs a strip of cross-ruled paper under the pen, and thus a continuous line is traced on the paper. The accompanying record extends from 2 P.M. of a certain Wednesday until 10.30 A.M. of the Friday following. The numbers 40, 50, 60, 70, 80, and 90 denote degrees Fahrenheit. There are 5 spaces from 50° to 60°. Hence one space corresponds to 2 degrees. The numbers 2, 4, 6, 8, and 10 indicate the time of day. Whether this is A.M. or P.M. can be determined by noting the position of these numbers with reference to the heavy curved lines marked NOON. The point *A* on the graph informs us that at 2 P.M. Wednesday the temperature was 80°. The point *B* between 6 P.M. and 7 P.M. Wednesday marks the highest temperature recorded.



The point *C* tells us that the temperature was about 65° at 6 A.M. Thursday.

The preceding record was made indoors, and the sudden fall from *D* to *E* was caused by the opening of a door leading into a cold hallway. The portion of the graph from *D* to *E* shows that the temperature of the room fell approximately 18 degrees in about 30 minutes.

ORAL EXERCISES

By reference to the graph (p. 200) answer the following:

1. With what temperature does the record begin? end?

2. What is the highest temperature recorded? the lowest?

3. About what time was the highest temperature recorded? the lowest?

4. How often did the instrument record a temperature of 82°? 76°? 73°? 68°?

5. At what times did it record a temperature of 82°? 76°? 73°? 68°?

6. To what practical use can a graph such as the one here explained be put?

90. Related pairs of numbers. A relation between two sets of numbers not necessarily connected with physical quantities such as temperature and time can also be expressed by a graph, as will be shown later.

The question What two numbers added give four? may be expressed by the equation $x+y=4$. Here x and y are any two numbers whose sum is 4. It can be seen by inspection that if $x=2$, $y=2$; if $x=1$, $y=3$; and if $x=0$, $y=4$, etc. Or we may assign to x any value, say -2; then the equation becomes $-2+y=4$, whence $y=6$.

In this manner we may obtain an unlimited number of sets of related values for x and y, some of which are given in the following table:

$$x + y = 4$$

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
If $x =$	2	1	0	3	4	-1	-2	5	6
then $y =$	2	3	4	1	0	5	6	-1	-2

91. Definitions and assumptions. In constructing the graph of an equation in two variables a number of assumptions must be made. These assumptions and some necessary definitions are now stated. It is agreed:

I. To have two lines at right angles to each other, as $X'OX$, called the **x-axis**, and $Y'OY$, called the **y-axis**, as in the figure on page 203.

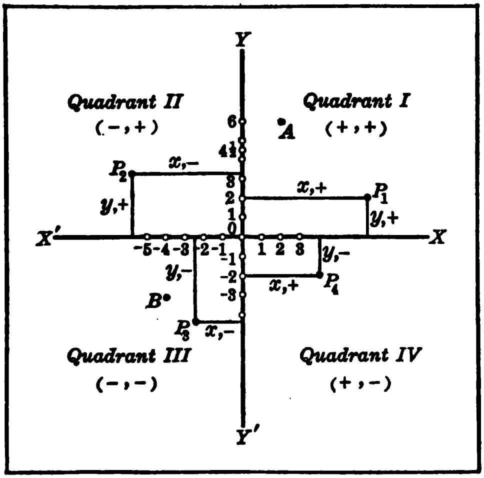
II. To have a line of definite length as a unit of distance. Then the number 2 will correspond to a distance twice the unit in length, the number $4\frac{1}{2}$ to a distance of $4\frac{1}{2}$ times the unit, etc.

III. That the distance (measured parallel to the x-axis) from the y-axis to any point in the surface of the paper be the **x-distance** (or abscissa) of that point, and the distance (measured parallel to the y-axis) from the x-axis to the point be the **y-distance** (or ordinate) of the point.

IV. That the x-distance of a point to the *right* of the y-axis be represented by a *positive* number, and the x-distance of a point to the *left* by a *negative* number; also that the y-distance of a point *above* the x-axis be represented by a *positive* number, and the y-distance of a point *below* the x-axis by a *negative* number. Briefly, ***distances measured from the axes to the right or upward are positive, to the left or downward are negative.***

V. That every point in the surface of the paper corresponds to a *pair of numbers*, one or both of which may be positive, negative, integral, or fractional.

VI. That of a given pair of numbers the first be the measure of the x-distance, and the second the measure of the y-distance. Thus the point (2, 6), or A in the figure, is the point whose x-distance is 2 and whose y-distance is 6. Again the point $(-4, -3)$, or B, is the point whose x-distance is -4 and whose y-distance is -3.



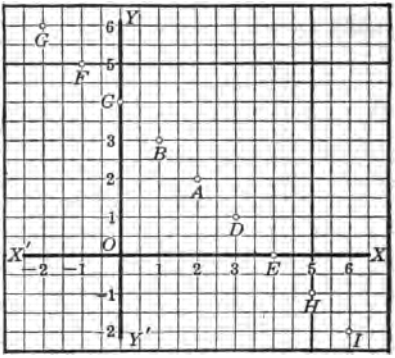
The point of intersection of the axes is called the **origin**.

The values of the x-distance and the y-distance of a point are often called the **coördinates** of the point.

Though not an absolute necessity, cross-ruled paper is a great convenience in all graphical work. Excellent results, however, can be obtained with ordinary paper and a rule marked in inches and fractions of an inch for measuring distances. Hence the graphical work which follows should not be omitted even though it is found inconvenient to obtain cross-ruled paper for class use.

EXAMPLE

Using $\frac{1}{6}$ inch for the unit of measure, locate points corresponding to the pairs of related numbers which follow: A, (2, 2); B, (1, 3); C, (0, 4); D, (3, 1); E, (4, 0); F, (–1, 5); G, (–2, 6); H, (5, –1); and I, (6, –2).



Solution. According to VI, page 203, the x-distance of the point (2, 2) is 2 and its y-distance is 2. Hence to locate point A at (2, 2) we measure, according to IV, two units to the right of the origin on the x-axis and from that point two units upward.

To locate point F at (–1, 5) we measure on the x-axis one unit to the left of the origin and from that point upward five units. Point F thus located corresponds to (–1, 5).

Points for the other pairs of numbers given in the example should be located by the pupil. The correct positions for these points can be seen in the figure.

Locating points as in the example above is called **plotting points.**

EXERCISES

Draw two axes and plot the following points, using $\frac{1}{2}$ inch as the unit of measure.

1. (2, 3); (3, 1); (4, 2.5); and (1.5, 4).

2. (4, –2); (5.5, –3); (6, –1); (–1.5, 4); (–1.8, 2); and (–3, –3).

3. (3, 0); (5, 0); (1, 0); (0, 0); (0, 3); and (0, 4.5).

4. (–2, 0); (–5, 0); (–1, 0); (0, –3.5); and (0, –4).

5. If the x-distance of a point is zero, where is the point located? Where is it located if both of its coördinates are zero?

92. The graph of an equation. On page 202 we computed several sets of values of x and y for the equation $x+y=4$. Later these points were plotted in locating A, B, C, D, E, F, G, H, and I of the example on the preceding page. It is evident from an inspection of their position that a straight line can be made to pass through all of the points there located. The line drawn through these points is said to be the **graph of the equation** $x+y=4$.

EXERCISES

1. Find and tabulate six pairs of values of x and y which satisfy the equation $2x+y=8$. Draw two axes and, using $\frac{1}{2}$ inch as the unit distance, plot each of the points. Are the six points in a straight line? Where do all the points lie whose x- and y-distances satisfy the equation $2x+y=8$? What, then, is the graph of the equation $2x+y=8$? Does $x=3$, $y=3$ satisfy this equation? Plot the point (3, 3). Is it on the graph of the equation? If the x- and y-distances of a point satisfy the equation $2x+y=8$, where is the point located? If the x- and y-distances of a point do not satisfy the equation $2x+y=8$, where is the point located?

Find and tabulate six pairs of values for x and y which satisfy each of the following equations. Use numbers not greater than 10. Use at least one negative value for x and one negative value for y. Then plot the six corresponding points. Can a straight line be drawn through the six points located for each exercise?

2. $2x+3y=6$.
4. $x+y=0$.
6. $2x=y$.
3. $4x-3y=12$.
5. $x-y=0$.
7. $y=3x$.

The preceding work should *convince* the student that the graph of an equation of the first degree in x and y is a *straight line.* This fact can be proved, but the student would not understand the proof were it given at present. Therefore it will be *assumed* that the graph of every linear equation in two variables is a straight line. And as a straight line is determined by *any two* of its points, in graphing a linear equation in two variables it will be sufficient to plot *any two points* whose x- and y-distances satisfy the equation, and then to draw through these two points a straight line. The two points most convenient to plot are usually the two in which the line cuts the axes. Occasionally these points come very close together, and consequently they will not determine accurately the position of the line. In such cases one should decide on two values of x rather far apart (such as 0 and 5, or 0 and -5) and compute the corresponding values of y. Two such points will fix the position of the line with sufficient accuracy.

If a line goes through the origin (as in Exercise 6 preceding), $x = 0$, $y = 0$ will do for one point, but a point not on the axes must be taken for the second one.

The essentials of the method of graphing a given linear equation in x and y are illustrated in the following example.

EXAMPLE

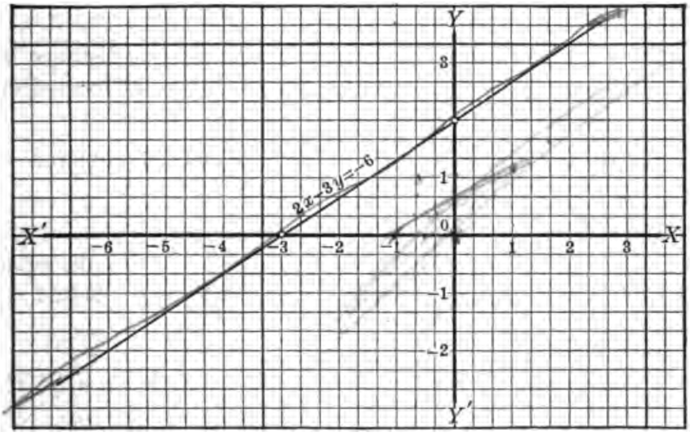
Graph the equation $2x - 3y = -6$. In this equation, if $x = 0$, $y = 2$; and if $y = 0$, $x = -3$. Here the point $(0, 2)$ is on the y-axis in the adjacent figure, 2 units above the origin; and the point $(-3, 0)$ is on the x-axis, 3 units to the left of the origin. The straight line through these two points is the graph of $2x - 3y = -6$.

The necessary work may be tabulated as follows:

$$2x - 3y = -6.$$

2x - 3y - 6

If $x =$	0	-3	2
then $y =$	2	0	$3\frac{1}{3}$



Check. If an error has been made in obtaining the value of x or y from the equation, or in plotting the values found, it can be quickly detected by plotting a third point, as $(2, 3\frac{1}{3})$, the values of whose x- and y-distances satisfy the equation. If this third point lies on the line determined by the first two points, the line has been correctly located; if it does not, an error has been made.

EXERCISES

Graph the following linear equations:

1. $x + y = 5.$
2. $x - y = 4.$
3. $x + 2y = 6.$
4. $4x + 3y = 12.$
5. $5x - 3y = 15.$
6. $2x + 3y = 10.$
7. $2x - y = 0.$
8. $x - 3y = 0.$
9. $x = 4.$

HINT. The equation $x = 4$ is equivalent to the equation $x + 0y = 4$. This last is satisfied by $x = 4$ and *any* value of y. Thus the pairs of values

(4, 3); (4, 6); ~~(4, 0)~~; (4, − 2), etc., satisfy the equation $x + 0y = 4$. Plotting these points, it is evident that the required graph is a line parallel to the y-axis and 4 units to the right of it.

10. $x = -4$. 12. $y = -3$. 14. $x = 0$.

11. $y = 6$. 13. $y = 0$. 15. $x = 5$.

16. If a point is on a line, do the values of its x-distance and its y-distance satisfy the equation of the line?

17. If the values of the x-distance and the y-distance of a point satisfy the equation of a line, is the point located on the graph of the equation?

18. Is the point (4, 3) on the line whose equation is $2x - 3y = 12$? is (0, 6)? is (6, 0)?

19. Can you determine without reference to the graph itself whether the point (6, 5) is on any of the graphs of the equations in Exercises 1–9 above? If so, on which does it lie?

It should now be clear that:

The equation of a line is satisfied by the values of the x-distance and the y-distance of any point on that line.

Any point the values of whose x-distance and whose y-distance satisfy the equation is on the graph of the equation.

93. Graphical solution of linear equations in two variables. If we construct the graphs of the two equations $5x + 4y = 12$ and $x - 2y = 8$ as indicated in the figure on page 209, it is seen that for the point of intersection of the graphs x is 4 and y is -2. Since the point $(4, -2)$ is on both graphs, these values should satisfy both equations. Substituting 4 for x and -2 for y in each equation, we get the identities $20 + (-8) = 12$, and $4 - (-4) = 8$. Thus the **graphical solution** of two linear equations consists in plotting the two equations and finding from the graph the value of x and the value of y at the point of intersection.

$y+6=0$

Since two straight lines *can intersect in but one point*, there can be but *one pair of values of x and y* which satisfies a pair of linear equations in two variables.

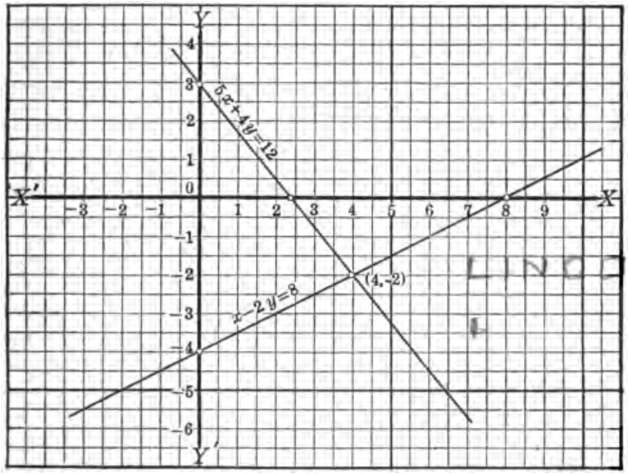
The necessary work is tabulated as follows:

$5x + 4y = 12$

If $x =$	0	$2\frac{2}{5}$	5
then $y =$	3	0	$-3\frac{1}{4}$

$x - 2y = 8$

If $x =$	0	8	2
then $y =$	-4	0	-3



LN

EXERCISES

Solve graphically the following pairs of linear equations and verify by substituting in each pair of equations the x and y values of the point of intersection as obtained from their graphs:

1. $x + y = 5,$
$2x + y = 8.$

2. $x + y = 7,$
$x - y = 3.$

3. $3x + y = 6,$
$x - 3y = -8.$

4. $x + y = 1,$
$2x - y = 8.$

5. $3x - 2y = 10,$
$x + 2y = 6.$

6. $x - 3y = 3,$
$2x + y = -8.$

7. $4x - y = 6,$
$2x + y = 9.$

8. $2x + 5y = 10,$
$x + y = 5.$

9. $x - 2y = 8,$
$2x - y = 4.$

10. $x + 2y = 4,$
$x + 2y = 6.$
omit

11. $x - 2y = 6,$
$2x - 4y = 9.$

12. $2x - y = 0,$
$x = 3y.$

94. Graphical representation of statistics. Scientific data and numerical statistics from the business world are frequently exhibited with striking clearness and brevity by means of graphs. The form of the graph obtained in any case depends on the character of the relation between the plotted numbers. Sometimes the resulting graph is a smooth curve, and then again it may be an irregular continuous line made up of straight lines of various lengths.

The following graphs are types which occur with increasing frequency in the magazines and in the daily papers. Such graphs display data effectively, and inferences not otherwise apparent can often be drawn from them.

BIOGRAPHICAL NOTE. *René Descartes.* One of the two or three most important advances ever made in mathematics was the discovery that algebraic equations could be represented geometrically. This great discovery was made by René Descartes (1596–1650), the French philosopher. Though never rugged in health, he took part in several campaigns when a young man, and it is said that during a weary winter spent in camp in Austria he first conceived the ideas that resulted in this important work. Though his writings read very differently from a modern book on the same subject, yet he developed all of the essentials of graphical representation. He saw that a letter, that is, a coördinate, might represent either a positive or a negative number, and so enforced upon mathematicians the conviction that negative integers are indeed numbers and that they are useful in algebraic operations. After his time they were not usually ruled out as absurd or impossible, as was commonly the case before. He also introduced the modern exponential notation, though he did not use negative or fractional exponents. To Descartes is due the use of the last letters of the alphabet for the unknown and the first letters for the known numbers. Thus he would have written the equation $x^3 - 8x + 16 = 40$ in the form $x^{3*} - 8x + 16 \infty 40$. Though the sign $=$ was used long before his time, he did not accept it. The asterisk he used to indicate that a certain power of the variable was lacking.

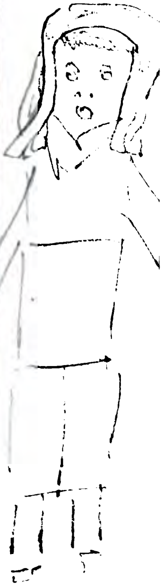
x + 2y = 4
x + 2y = 6
2x = 10 - 4y
2x = 13 - 4

6 = 10 - 4y
- 4
2x = 10 - 5

2' = 10 -

René des Cartes Philosophe
et Gentilhomme François né en Touraine et decedé en Suede l'an 1650 agé de 54 ans.
Definir la Nature et de l'ame et du Corps,
De la Terre et du Ciel decouvrir les ressorts,
D'un être Souverain demontrer l'existence,
Voila quel fut l'employ de René du Perron:
Et luy seul eut plus de Science
Que n'en eurent ensemble Aristote et Platon.

RENÉ DESCARTES



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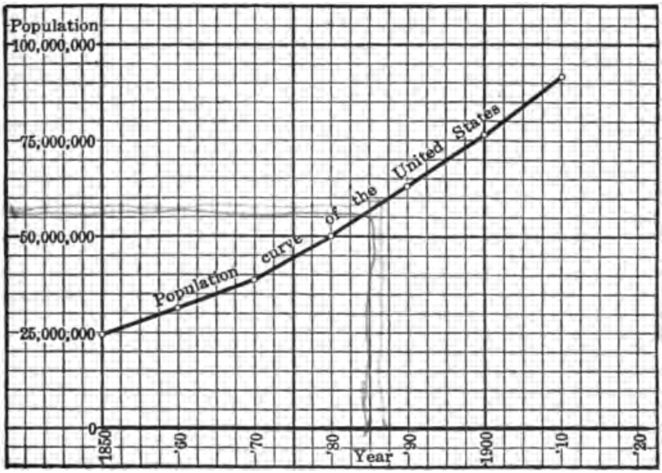
EXAMPLE 1

The census reports of the United States show that the population in millions for ten-year intervals was as follows:

Year	1850	1860	1870	1880	1890	1900	1910
Population in millions	23.2	31.4	38.6	50.2	62.6	76.0	92.0

The graphical representation of these statistics is given below. On the graph the population is measured parallel to the vertical axis, $\frac{1}{10}$ of an inch representing 5,000,000 people. The ten-year intervals are measured along the horizontal axis, with $\frac{1}{10}$ of an inch representing $2\frac{1}{2}$ years.



QUESTIONS ON EXAMPLE 1

1. What population does the graph indicate for 1885? for 1905?

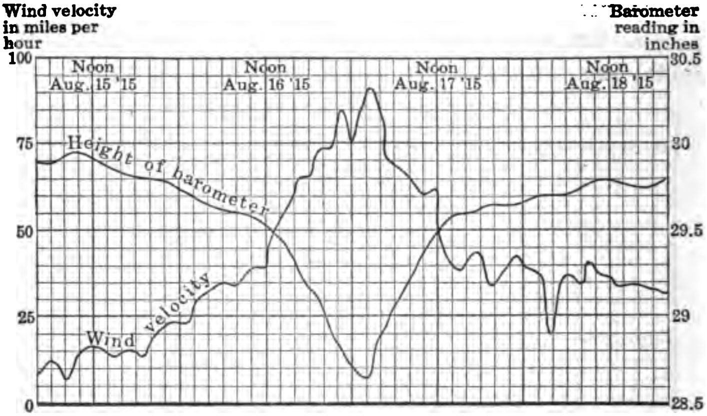
2. What cause may be assigned for the downward bend of the curve at 1870?

3. What population does the graph indicate for the United States in 1915? in 1920?

4. What causes might make the population of the United States in 1920 differ from the value indicated by the curve?

EXAMPLE 2

Graphical record of atmospheric pressure and wind velocity for the Galveston, Texas, storm period of August 15–18, 1915. The great storm occurred August 17.



QUESTIONS ON EXAMPLE 2

1. What was the velocity of the wind at midnight August 15? at midnight August 16?

2. At what times was the wind velocity 40 miles per hour? 80 miles per hour?

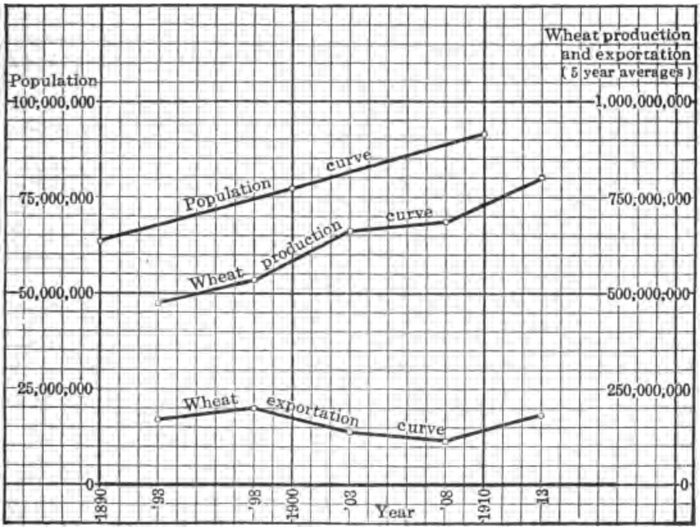
3. What was the reading for barometric pressure (height of the barometer) at midnight August 15? at midnight August 16? at 2 A.M. August 17?

4. What was the lowest reading of the barometer and the highest velocity of the wind? When did this occur?

EXAMPLE 3

In the following graph the upper curve shows in millions the population of the United States for 1890, 1900, and 1910. The middle and lower curves show the average yearly production and exportation respectively of wheat in millions of bushels (using the average for 5-year intervals) for the years 1891–1915 inclusive. The general tendency of production and exportation is made more evident by plotting not yearly values but averages for five-year intervals, since by this method the high or low record for any one year is not emphasized unduly.

Intervals	1891–1895	1896–1900	1901–1905	1906–1910	1911–1915
Average yearly production	477	540	660	681	803
Average yearly exportation	167	197	140	116	188



QUESTIONS ON EXAMPLE 3

1. Estimate the population of the United States for 1892; for 1907; for 1912.

2. Compare the rate of increase in the production of wheat since 1901 with the rate of increase of the population.

3. Does the graph reveal any general tendency in the number of bushels of wheat which is exported? What tendency do you observe?

4. Does the graph indicate that the number of bushels exported will increase to any considerable extent? Explain.

EXERCISES ON THE GRAPHING OF STATISTICS

1. The records of the United States Weather Bureau of a certain city show hourly temperature records for a certain day in July. These records reported for the hours ending 1 A.M. to 12 P.M. inclusive, expressed in degrees, are as follows:

A.M., hour ending at	1	2	3	4	5	6	7	8	9	10	11	12
Temperature record .	84	83	80	80	79	77	77	79	84	88	89	93

P.M., hour ending at	1	2	3	4	5	6	7	8	9	10	11	12
Temperature record .	98	100	88	72	81	86	84	82	80	75	74	74

Graph the foregoing data.

A thunderstorm occurred on the day in which the above record was made. From the graph determine the time of the storm. How do you account for the sudden fall in temperature after 2 P.M.? for the sudden rise in the graph after 4 P.M.?

2. From reports gathered from several cities it has been estimated that for 100 children in school at the age of 8 the numbers of pupils remaining in school (not counting those eliminated by death) for each year up to and including age 18 are as follows:

Age of pupils . .	8	9	10	11	12	13	14	15	16	17	18
Number in school .	100	100	100	98	97	88	70	47	30	16.5	8.6

Graph the foregoing data. Let $\frac{1}{2}$ inch represent one year on the horizontal axis and ten pupils on the vertical axis.

From the graph determine the years for which the tendency to leave school first becomes pronounced. Does the graph show a greater tendency on the part of 14-year-old and 15-year-old pupils to leave school than on the part of those of 12 and 13 years? How does the graph show this fact? Explain.

3. Estimates worthy of consideration are here given which show a contrast in the average weekly earnings for boys who leave school at the age of 14 (the usual end of the grammar-school period) with weekly earnings of boys who leave school at the age of 18 (the usual end of the high-school period)

Age	Boys who leave school at the age of 14 earn for the years indicated at the left the weekly wages given below:	Boys who leave school at the age of 18 earn for the years indicated at the left the weekly wages given below:
14	$4.00	0
16	$5.00	0
18	$7.00	$10.00
20	$9.50	$15.00
22	$11.00	$20.00
24	$12.00	$24.00
25	$13.00	$30.00

Graph the two sets of data on the same axes.

Determine from the graph which group of boys receives the more rapid increase in weekly wage.

4. The value in millions of dollars of the merchandise imported into and exported from the United States for the years 1909 to 1916 inclusive was as follows:

Year	1909	1910	1911	1912	1913	1914	1915	1916
Imports	1311	1556	1527	1653	1813	1893	1674	2197
Exports	1663	1744	2049	2204	2465	2364	2768	4333

Plot the foregoing data, using $\frac{1}{2}$ inch for one year on the horizontal axis and for 250 million dollars on the vertical axis.

How do you account for the decreased imports of 1915? the decreased exports of 1914? the increased exports of 1915?

5. The purchasing power of a dollar in 1896 is compared to the purchasing power of a dollar for the years 1897 to 1915, respectively, in the following table:

Year	1896	1897	1898	1899	1900	1901	1902	1903	1904	1905	1906	1907	1908	1909	1910	1911	1912	1913	1914	1915	1916
Cents	100	98.4	95.2	83.9	79.5	74.5	70.2	72.7	72.7	68.6	65.6	65.6	66.7	60	57.8	58.8	56.9	57.4	57.7	56.6	

Graph the foregoing data, showing the change in the purchasing power of a dollar through the given interval.

What tendency in the change in the purchasing power of a dollar does the graph indicate? Could more or less merchandise be bought for a given sum in 1900 or in 1910? Approximately how much more? A salary of $1500 in 1910 was equivalent to what salary in 1896?

CHAPTER XVIII

LINEAR SYSTEMS

95. Definitions. A simple or **linear** equation in several unknowns is one which may be put in such a form that

(*a*) no unknown appears in any denominator;

(*b*) only one unknown appears in any term;

(*c*) only the first power of any unknown is involved.

The following equations are linear: $2x+3y=1$; $4u+6v+2w=4$; the following equations are not linear: $2x+3xy-4y=1$; $4/u-6/v+2/w=2$; $x^2+3x-4y=(x+2)(2y-3)$.

In a previous chapter we have seen that a simple equation in one unknown has only one root; that is, the value of the unknown in such an equation is a constant.

Thus the value of the unknown in the equation $3x+4=7$ is the number 1, and this is the only root of the equation.

A linear equation in two unknowns is satisfied by an unlimited number of pairs of values for the two unknowns. A single equation in more than one unknown is called an **indeterminate** equation. The unknowns are sometimes called variables.

Thus the equation $x+y=10$ is satisfied by any pair of numbers whose sum is 10. Evidently, if one includes positive, negative, and fractional numbers, there is no limit to the number of pairs whose sum is 10. (See page 202.)

Two or more equations involving two or more unknowns are called a **system** of equations.

A system of equations all of which are satisfied by the same values of the unknowns is called a **simultaneous system** of equations.

The two equations $x + y = 9$ and $x - y = 1$ form a simultaneous system. The two numbers that satisfy both equations must be such that their sum is 9 while their difference is 1. These conditions are satisfied by $x = 5$, $y = 4$.

A set of values (one for each unknown) which satisfies an equation in two or more unknowns is sometimes called a solution of the equation; and a set which satisfies a system is often called a solution of the system. In this book, however, the word *solution* will be used to denote the *process of solving* either a single equation or a system. The values of the unknown which satisfy an equation in one unknown will be called *roots*, and a set of values for the unknowns satisfying an equation in two or more unknowns, or a system of such equations, will be called a **set of roots.**

ORAL EXERCISES

In Exercises 1–4 find the value of x corresponding to each of the values for y indicated at the right:

1. $x + y = 10$. $y = 1, 5, 6$.
2. $x - y = 1$. $y = 4, 10, 0$.
3. $x + 2y = 16$. $y = 0, 4, 8$.
4. $x - 2y = 0$. $y = 1, 4, -1$.

In Exercises 5–8 determine which of the pairs of numbers written at the right of each equation satisfies that equation. (The first number of a pair always denotes the value of x.)

5. $2x + 3y = 10$ $(2, 2)$; $(1, 3)$; $(8, -2)$.
6. $3x - y = 0$ $(2, 4)$; $(4, 2)$; $(-1, 3)$.

7. $y - x = 0$ (3, 4); (6, 6); (−1, −1).

8. $3y - 6 = 2x$ (1, 4); (−3, 0); (−3, −1).

In Exercises 9–12 find two pairs of numbers which satisfy each equation, and two other pairs which do not.

9. $x + 2y = 7.$

10. $x - 3y = 5.$

11. $2x + 3y = 16.$

12. $3y + 4x = 2.$

96. Solution by addition and subtraction. It was shown in the previous section that there is an unlimited number of sets of roots of a given linear equation in x and y, and that there is also an unlimited number of pairs of values of x and y which do not satisfy the equation. We now proceed to give an algebraic method of determining whether or not there is any common set of roots for two given linear equations. It turns out that there is usually one and only one such set of roots. This set may always be found by the method illustrated in the following examples.

EXAMPLES

1. Solve the system $\begin{cases} x + y = 4, & (1) \\ x - 2y = 1. & (2) \end{cases}$

Solution. Eliminate x first, thus:

(1) − (2), $3y = 3.$ (3)

(3) ÷ 3, $y = 1.$ (4)

Substituting 1 for y in either (1) or (2), say in (1),

$$x + 1 = 4. \qquad (5)$$

Solving (5), $x = 3.$ (6)

Check. Substituting 3 for x and 1 for y in (1) and (2) gives the identities

$$3 + 1 = 4,$$

and

$$3 - 2 = 1.$$

2. Solve the system $\begin{cases} 5x + 3y = 1, & (1) \\ 3x - 2y = 12. & (2) \end{cases}$

Solution. Eliminate y first, thus:

(1) · 2, $10x + 6y = 2.$ (3)

(2) · 3, $9x - 6y = 36.$ (4)

(3) + (4), $19x = 38.$ (5)

(5) ÷ 19, $x = 2.$ (6)

Substituting 2 for x in (2), $6 - 2y = 12.$ (7)

Solving (7), $y = -3.$ (8)

Check. Substituting 2 for x and -3 for y in (1) and (2) gives

$$10 - 9 = 1,$$

and

$$6 + 6 = 12.$$

Either x or y could have been eliminated first. The multipliers necessary to eliminate x are 3 and 5, while the multipliers necessary to eliminate y are the *more convenient numbers* 2 and 3.

When the notation (3) – (4) is used in a solution, it indicates the subtraction of the first member of equation (4) from the first member of equation (3), the subtraction of the second member of equation (4) from the second member of equation (3), and the writing of the two results as an equation. The process of adding the corresponding members of the two equations is indicated by writing (3) + (4).

The notation (3) · 6 indicates that both members of equation (3) are multiplied by 6, and (3) ÷ 6 indicates that both members of equation (3) are divided by 6.

With the meanings just explained it is customary to speak of the addition or the subtraction of two equations and of the multiplication or division of an equation by a number.

The method of the preceding solutions is stated in the

Rule. *If necessary, multiply the first equation by a number and the second equation by another number, such that the coefficients of the same unknown in each of the resulting equations will be numerically equal.*

If these coefficients have like signs, subtract one equation from the other; if they have unlike signs, add. Then solve the equation thus obtained.

Substitute the value just found, in the simplest of the preceding equations which contains both unknowns, and solve for the other unknown.

Check. *Substitute for each variable in the original equations its value as found by the rule. If the resulting equations are not obvious identities, simplify them until they become such.*

An attempt to solve by the rule the pair

$$\begin{cases} 3x - 6y = 40, & (1) \\ x - 2y = 8, & (2) \end{cases}$$

gives

$$3x - 6y = 40, \quad (3)$$

$$3x - 6y = 24. \quad (4)$$

(3) − (4), $0 = 16$, which is false.

This result indicates that (1) and (2) do not form a simultaneous system, but are **incompatible** equations.

The graphs of a pair of incompatible linear equations are parallel lines (see Exercise 10, p. 210).

An attempt to solve by the rule the system $\begin{cases} x + 2y = 8, \\ 3x + 6y = 24, \end{cases}$ gives $0 = 0$. Here the second equation divided by 3 gives the first. Therefore any set of roots of the first is a set of the second. If we choose to regard the two equations as really different, which is not at all necessary, we say that they have an infinite (unlimited) number of sets of roots. Two or more equations having this property constitute **an indeterminate system.**

EXERCISES

Solve the following systems and check results:

1. $x + y = 3,\quad x - y = 1.$
2. $x + y = 7,\quad x - y = 1.$
3. $2x + y = 9,\quad x - y = 6.$
4. $x + y = 5,\quad x - 3y = 1.$
5. $x + y = -3,\quad x - y = 3.$
6. $x + 2y = 7,\quad 5x - 2y = 11.$

7. $2y + x = 4,\quad 3y - x = 21.$

8. $7r - s = 2,\quad 6r - s = 3.$

9. $3x - y = 3,\quad 5x + 2y = 16.$

10. $4x + y = 2,\quad x - 2y = 5.$

11. $x + 2y = 1,\quad 3x + 10y = 4.$

12. $5m - 3n = 0,\quad 15m + 12n = 75.$

13. $x - 6y = 7,\quad 12y - x = -1.$

14. $2x + 25y = 70,\quad 3x = 10y + 10.$

15. $4p + q = 5,\quad p - 4q = 14.$

16. $12x + 5y = 6,\quad 3x - 3y = 10.$

17. $3x - 4y = 14,\quad 3y - 4x = -14.$

18. $4x + 3y = 5,\quad 9y - 8x = 0.$

19. $6u + 8v = 26,\quad 5u - 3v = 70.$

20. $11x + 7y = 111,\quad 3y - 4x = 4.$

21. $5y + 3z = 37,\quad 9z + 15y = 111.$

97. Solution by substitution. The method of solving a system of two linear equations by substitution is illustrated in the following example.

EXAMPLE

Solve the system $\begin{cases} 4x - 12y = 44, & (1) \\ 8x + 11y = 18. & (2) \end{cases}$

Solution. From (1), $$4x = 12y + 44, \tag{3}$$

or $$x = 3y + 11. \tag{4}$$

Substituting $3y + 11$ for x in (2),

$$8(3y + 11) + 11y = 18. \tag{5}$$

Simplifying, $$24y + 88 + 11y = 18. \tag{6}$$

Collecting, $$35y = -70, \tag{7}$$

or $$y = -2. \tag{8}$$

Substituting -2 for y in (4),

$$x = -6 + 11 = 5. \tag{9}$$

Check. Substituting 5 for x and -2 for y in (1) and (2) gives

$$20 + 24 = 44,$$

and $$40 - 22 = 18.$$

The method of the preceding solution is stated in the

Rule. *Solve either equation for one unknown in terms of the other.*

Substitute this value in place of the unknown in the equation from which it was not obtained and solve the resulting equation.

Substitute the definite value just found, in the simplest of the preceding equations which contains both unknowns, and solve, thus obtaining a definite value for the other unknown.

Check. See page 221.

The method of substitution emphasizes the fact that the values of x and y which are sought are the same in both equations. Hence an expression for an unknown obtained from one equation is substituted for that unknown in the other equation. This method is useful when one of the unknowns can be expressed in terms of the other without using fractions or when simple fractions only are involved.

EXERCISES

Solve by the method of substitution and check results:

1. $x - 2y = 8,$
$3x + 2y = 8.$
2. $x - 2y = -1,$
$4x - y = 10.$
3. $14m - 2n = 1,$
$n - 6m = 0.$
4. $6x + 10y = 42,$
$2y = 3x.$
5. $5x + 3y = -1,$
$2x - 6y = 8.$
6. $x + y = 7,$
$2x + 3y = 17.$
7. $3x = 6y - 3,$
$y + 4x = 5.$
8. $5x + 10y = 25,$
$5x - 3y = 9.$
9. $20y - 3z = 1,$
$z - 6y = 0.$
10. $3x + 12 = 3 + y,$
$y = x + 1.$
11. $18 + 2k = h,$
$h + k = -9.$
12. $2x = 4y + 6,$
$7x + 3y = 4.$

98. Simultaneous equations containing fractions. The method of solving a system of two linear equations containing fractions is illustrated by the following example.

EXAMPLE

Solve the system
$$\begin{cases} \dfrac{8x}{3} - \dfrac{5}{6} = \dfrac{3y}{2}, & (1) \\ \dfrac{2x}{3} = \dfrac{y}{4} + \dfrac{7}{12}. & (2) \end{cases}$$

Solution. (1) · 6, $16x - 5 = 9y.$ (3)

Transposing in (3), $16x - 9y = 5,$ (4)

(2) · 12, $8x = 3y + 7.$ (5)

Transposing in (5), $8x - 3y = 7.$ (6)

The system (4) and (6) can now be solved by addition and subtraction.

As in the foregoing solution, it is usually best to clear the equations of fractions and write them in the form of (4) and (6) before attempting to eliminate one of the unknowns. Equations (4) and (6) are each in what is called the **general form** of a linear equation in two unknowns. This form is represented for all such equations by $ax + by = c$, where a, b, and c denote numbers, or known literal expressions.

EXERCISES

Solve the following systems and check results:

1. $4x + \dfrac{2y}{3} = \dfrac{26}{3},\quad 3y - \dfrac{7x}{2} = -4.$

2. $\dfrac{5x}{6} + \dfrac{y}{4} = 7,\quad \dfrac{2x}{3} - \dfrac{y}{8} = 3.$

3. $x - \dfrac{3y}{5} = \dfrac{18}{5},\quad 7x + \dfrac{8y}{3} = -16.$

4. $$\begin{aligned} .4x + .9y &= 5.7, \\ x - \frac{y}{2} &= \frac{1}{2}. \end{aligned}$$

5. $$\begin{aligned} \frac{x}{3} + \frac{y}{2} &= \frac{13}{6}, \\ 2y - 3x &= 0. \end{aligned}$$

6. $$\begin{aligned} \frac{x}{3} + \frac{y}{4} &= -\frac{7}{8}, \\ \frac{y}{3} - \frac{x}{4} &= \frac{11}{12}. \end{aligned}$$

7. $$\begin{aligned} .04x + .3y &= 1, \\ .5x - .25y &= 4.5. \end{aligned}$$

8. $$\begin{aligned} 7x - 4y &= 14, \\ \frac{2x+7}{2} - y &= 4. \end{aligned}$$

9. $$\begin{aligned} \frac{x}{3} + \frac{y}{4} &= \frac{1}{6}, \\ \frac{x-2}{5} - \frac{y+1}{2} &= \frac{1}{2}. \end{aligned}$$

10. $$\begin{aligned} \frac{2x+6}{5} + \frac{y+4}{5} &= 3, \\ \frac{7x+1}{3} - \frac{11y-4}{7} &= 4. \end{aligned}$$

11. $$\begin{aligned} \frac{x+y}{2} - 8 &= \frac{x-y}{3}, \\ \frac{x-y}{4} + \frac{x+y}{3} &= 11. \end{aligned}$$

12. $$\begin{aligned} \frac{x}{6} - \frac{y}{4} &= 1, \\ \frac{4+5x}{11} - \frac{3-2y}{5} + 4 &= 0. \end{aligned}$$

13. $$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{5}{6}, \\ \frac{1}{x} - \frac{1}{y} &= \frac{1}{6}. \end{aligned}$$

Hint. Solve without clearing of fractions, using addition and subtraction.

14. $$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{4}{15}, \\ \frac{3}{x} + \frac{2}{y} &= \frac{19}{15}. \end{aligned}$$

15. $$\begin{aligned} \frac{6}{x} + \frac{7}{y} + \frac{3}{2} &= 0, \\ \frac{7}{x} - \frac{6}{y} &= \frac{16}{3}. \end{aligned}$$

16. $$\begin{aligned} \frac{5}{x} + 12 &= 17, \\ \frac{2}{x} - 3y &= 0. \end{aligned}$$

17. $$\begin{aligned} \frac{x-y}{2} &= \frac{25}{6} - \frac{x+y}{3}, \\ \frac{x+y-9}{2} + \frac{x-y+6}{3} &= 0. \end{aligned}$$

18. $$\begin{aligned} \frac{x-2}{5} - \frac{y-10}{4} &= \frac{10-x}{3}, \\ \frac{y+2}{3} - \frac{2x+y}{16} &= \frac{x+13}{8}. \end{aligned}$$

19. $$\begin{aligned} \frac{x+2}{7} + 8 - 2x &= \frac{x-y}{4}, \\ 2y - \frac{3x-2y}{3} &= 3x + 4. \end{aligned}$$

20.
$$\frac{2}{x-y}-\frac{3}{2x+3y-14}=0, \qquad (1)$$
$$2x-\frac{y}{4}=\frac{5}{4}. \qquad (2)$$

Hints. Multiplying (1) by $(x-y)(2x+3y-14)$,

$$2(2x+3y-14)-3(x-y)=0, \qquad (3)$$

or
$$x+9y=28. \qquad (4)$$

(2) · 4,
$$8x-y=5. \qquad (5)$$

Now solve equations (4) and (5) by substitution and check as usual.

21.
$$\frac{4x+y}{6x+y}=\frac{2}{5},$$
$$x+\frac{10y}{3}=\frac{71}{3}.$$

22.
$$\frac{4x}{3y+2}=4,$$
$$\frac{2x}{4x-3y+1}=\frac{1}{4}.$$

23.
$$\frac{3x-4y+2}{x-y}=3,$$
$$\frac{x+y}{x}=\frac{5}{3}.$$

24.
$$\frac{x-3}{y+1}=\frac{x-6}{y-2},$$
$$\frac{x+y-5}{x-y+1}=3.$$

25.
$$\frac{5x+3y-5}{3}=y+\frac{5}{3},$$
$$\frac{25x}{10x+2y}=1.$$

26.
$$\frac{r+5}{r+1}=\frac{s+2}{s-2},$$
$$\frac{11r+4}{r+s+1}=4.$$

27.
$$\frac{2}{x+y+2}=\frac{1}{3x+5y-5},$$
$$\frac{x}{4}+\frac{y}{2}=1.$$

28.
$$\frac{4}{x-1}=\frac{3}{1-y},$$
$$\frac{5}{2y-5}=\frac{7}{2x-39}.$$

In the following problems the student should state *two* equations in *two* unknowns. Instead of using x and y, the *first letter* of the word denoting an unknown should be used to represent that unknown. Thus in Problem 5, p. 227, n should represent the number of nickels and q the number of quarters. The plan here suggested is desirable for many reasons, and should be followed in all problems containing two or more unknowns unless the words denoting two of the unknowns begin with the same letter.

In solving problems like 1–12 on pages 46–47, there are really two unknowns involved; but one of the equations to which each of those problems leads is so simple that what amounts to the method of substitution was employed by expressing one unknown in terms of the other.

PROBLEMS

1. The difference of two numbers is 20 and their sum is 36. Find the numbers.

2. The quotient of two numbers is 12 and their sum is 39. Find the numbers.

3. Find two numbers whose difference is 28 and whose quotient is 5.

4. The value of a certain fraction is $\frac{3}{4}$. If 6 be added to the numerator and 12 to the denominator, the value of the resulting fraction is $\frac{2}{3}$. Find the fraction.

5. A collection of nickels and quarters, containing 63 coins, amounted to $8.35. How many coins of each kind were there?

6. If $\frac{2}{3}$ be subtracted from the numerator and $\frac{4}{3}$ added to the denominator of a certain fraction, the value of the resulting fraction is $\frac{1}{4}$. The sum of the numerator and the denominator of the original fraction is 11. Find the fraction.

7. The difference between the numerator and the denominator of a certain proper fraction is 11. If $\frac{1}{2}$ be added to the numerator and $\frac{3}{4}$ be taken from the denominator, the value of the resulting fraction is $\frac{6}{19}$. Find the fraction.

8. Two weights balance when one is 12 inches and the other 8 inches from the fulcrum. If the second weight increased by 8 pounds is placed 6 inches from the fulcrum, the balance is maintained. Find the two weights.

9. Two weights balance when one is 10 inches and the other 8 inches from the fulcrum. If the first weight is decreased by 4 pounds, the other weight must be moved 2 inches nearer the fulcrum to balance. Find the weights.

10. In 20 years A will be twice as old as B. Ten years ago A was 8 times as old as B. Find the age of each now.

11. The perimeter of a rectangle is 184 feet and the length is 8 feet more than twice the width. Find the dimensions of the rectangle.

12. A part of $1500 is invested at 6% and the remainder at 5%. The combined yearly income is $81. Find the number of dollars in each investment.

13. A part of $3000 is invested at $4\frac{1}{2}$% and the remainder at $3\frac{1}{2}$%. The annual income from the $4\frac{1}{2}$% investment is $3 less than double the annual income from the $3\frac{1}{2}$% investment. Find the number of dollars in each investment.

14. A part of $5000 is invested at 4% and the remainder at 6%. The 4% investment yields $16 less in 4 years than the one at 6% does in 3 years. Find the number of dollars in each investment.

15. Five rubles are worth 5 cents less than 10 marks, and 12 marks are worth 4 rubles and a dollar. Find the value of a ruble and of a mark in cents.

16. During the war the value of marks in New York fell so much that on Jan. 1, 1916, 6 marks were worth exactly as much as 5 marks were worth on Aug. 1, 1914. Between the two dates mentioned the value of 10 marks decreased by 40 cents. Find the value of a mark, in cents, at each date.

17. The sum of the two digits of a two-digit number is 9. If 45 be subtracted from the number, the result will be expressed by the digits in reverse order. Find the number.

Solution. Let $t =$ the digit in tens' place,
and $u =$ the digit in units' place.
Then $t + u = 9.$ (1)

But t standing in tens' place has *its numerical value multiplied by* 10. Therefore the number is represented by the binomial $10t + u$, and the number formed by the digits in reverse order is represented by the binomial $10u + t$.

Hence $10t + u - 45 = 10u + t.$ (2)
Simplifying (2), $t - u = 5.$ (3)
Solving (1) and (3), $t = 7$, and $u = 2$.
Hence the number is 72.

Check. $7 + 2 = 9$, $72 - 45 = 27$.

18. The sum of the digits of a two-digit number is 10. If 54 be added to the number, the result is expressed by the digits in reverse order. Find the number.

19. The tens' digit of a two-digit number is half the units' digit. If 36 be added to the number, the result is expressed by the digits in reverse order. Find the number.

20. If a two-digit number be divided by the sum of its digits, the quotient is 4. Twice the given number is 9 greater than the number expressed by the same digits in reverse order. Find the number.

21. If a two-digit number be increased by 4 and then the result be divided by the sum of its digits, the quotient is 3. If twice the number be divided by the tens' digit, the quotient is 29. Find the number.

22. If a two-digit number be divided by the sum of its digits, the quotient is 7. If the number formed by the digits in reverse order be divided by 3 plus the sum of the digits, the quotient is 3. Find the number.

The **reciprocal** of a number is a fraction of which 1 is the numerator and the number itself is the denominator. Thus $\frac{1}{2}$ and $\frac{1}{a}$ are the reciprocals of 2 and a respectively.

23. What is the reciprocal of 6? $7a$? $\frac{2}{3}$? $\frac{3}{4x}$? $2\frac{1}{2}$? $3\frac{2}{5}$?

24. The sum of the reciprocals of two numbers is $\frac{11}{28}$, and the difference of their reciprocals is $\frac{3}{28}$. Find the numbers.

25. The difference of the reciprocals of two numbers is $\frac{1}{24}$. The quotient of the greater number divided by the less is $\frac{16}{15}$. Find the numbers.

26. If 5 grams be taken from one pan of a balance and placed in the other, the sum of the weights in the first will be $\frac{1}{2}$ the sum of those in the second. But if 15 grams be taken from the second and placed in the first, the weights in the two pans will then balance. Find the weight in each pan at first.

27. A gives B $30; then B has twice as much money as A. B then gives A $150 and has left $\frac{1}{5}$ as much as A. How many dollars had each at first?

28. The circumference of the fore wheel of a carriage is $1\frac{1}{2}$ feet less than that of the rear wheel. The fore wheel makes as many revolutions in going 286 feet as the rear wheel does in going 325 feet. Find the circumference of each wheel.

29. A and B together can do a piece of work in $7\frac{1}{5}$ days. They work together for 3 days, and A finishes the job by himself in 7 days. How many days would each require alone?

30. A man rows 10 miles downstream in 2 hours and returns in 4 hours. Find the rate of the river and his rate in still water.

HINT. Let $x =$ the man's rate in still water in miles an hour, and $y =$ the rate of the river in miles an hour. Then his rate downstream is $x + y$ miles an hour, and upstream $x - y$ miles an hour.

31. A boat goes downstream 45 miles in 3 hours and upstream 15 miles in 3 hours. Find its rate in still water and the rate of the current.

32. The rate of a boat in still water is $7\frac{1}{2}$ miles an hour. It goes down the river from A to B in 12 hours. It returns one half the distance from B to A in 9 hours. Find the rate of the river and the distance from B to A.

33. A boat which runs 12 miles an hour in still water goes downstream from A to C in 7 hours. It returns upstream to B, 36 miles below A, in 5 hours. Find the distance from A to C and the rate of the stream.

99. Literal equations in two unknowns. Linear systems in which the unknowns have literal coefficients are solved by the method of addition and subtraction.

EXERCISES

In the following exercises consider *a*, *b*, *c*, and *d*, as known numbers; solve for the other letters involved and check.

1. $x + y = 3a,\quad x - y = a.$

2. $5x + 4y = 17a,\quad 8x + y = 11a.$

3. $3x - y = 10b,\quad 4x + 9y = 3b.$

4. $5x - 4y = 10a - 4,\quad x - 2ay = 0.$

5. $3p + 4q - a = p - q + 22a,\quad p + a - q = 3p - 2q + 8a.$

6. $3x + 4y = 6c,\quad \dfrac{x}{c} - \dfrac{3y}{4c} = 2.$

7. $\dfrac{2x}{5} + \dfrac{2y}{3} = \dfrac{6a}{5},\quad \dfrac{y}{2} + \dfrac{x}{2} = a.$

8. $7.5m + 3n = 6b,\quad .25m + .5n = 0.$

9. $3x - 3y - 4a = x - b,\quad x + y + a + b = 2(a + b).$

10. $\dfrac{x}{2a} - \dfrac{2y}{a} = -10,\quad \dfrac{3x}{a} - \dfrac{7y}{4a} = \dfrac{3}{2}.$

11. $x + y = a,$
$x - y = b.$

12. $ax + 3y = 3 + 6a,$
$a^2x + y = 5a.$

13. $cx + y = 3,$
$c(y - 3) = x.$

14. $4x - 3y = 12(a - b),$
$3x - 2y = 9a - 8b.$

15. $4x + 2y = a,$
$2x + 4y = b.$

16. $2x - ay = b,$
$8x - by = 4a.$

17. $\frac{1}{x} - \frac{1}{y} = a,$
$\frac{3}{x} - \frac{2}{y} = b.$

18. $\frac{a}{x} + \frac{b}{y} = c,$
$\frac{a}{x} - \frac{b}{y} = d.$

19. $ax + by = a + b,$
$ax - by = a - b.$

20. $ax - by = c,$
$3ax - 2by = 4c.$

21. $ax + by = c,$
$hx + ky = m.$

GENERAL ORAL EXERCISES

1. If one book costs a dollars, what will $c + d$ books cost?

2. If a books cost b dollars, what will one book cost? c books?

3. What is the perimeter of a rectangle whose length is a and whose width is b? What is its area?

4. What is the perimeter of a rectangle whose length is $4a$ and whose width is b? What is its area?

5. The base of a triangle is $a + b$. The altitude is $a - b$. Find the area.

6. The base and the altitude of a triangle are each equal to $x - 2y$. Find the area.

7. The area of a triangle is k. The base is b. Find the altitude.

8. If x denotes A's age now, what does $x - 8$ denote? $x + 5$? What does the equation $x + 5 = 2(x - 8)$ signify?

9. Forty men pay d dollars each as dues to a society. The expenses of the society are n dollars. How many dollars are left in the treasury? How would you interpret the result if n is greater than $40\,d$?

10. If 10 apples can be bought for x cents, how many can be bought for y cents?

11. If a apples can be bought for 25 cents, how many can be bought for c cents?

12. A farmer has grain enough to last one horse d days. How long will this grain last k horses?

13. A farmer has grain enough to last h horses d days. How long will it last k horses?

GENERAL PROBLEMS

1. The altitude of a triangle is a inches and the base is 10 inches. If 2 inches be taken from the altitude, by how much must the base be increased so that the area will be the same as before?

HINT. Let $x =$ the increase of the base in inches.

Then $\frac{10a}{2} = \frac{(a-2)(10+x)}{2}$.

2. The altitude of a triangle is a feet, the base is b feet. The altitude is increased h feet and the base decreased so that the area is the same as before. How many feet are taken from the base?

3. The sum of two numbers is s and their difference is d. Find the numbers.

4. The first of two numbers is a times the second, and the first minus the second equals b. Find the numbers.

5. The sum of two numbers is b, and the quotient of the first divided by the second equals a. Find the numbers.

6. If a be added to the numerator of a certain fraction, the value of the resulting fraction is 2. If b be added to the denominator, the value of the resulting fraction is 1. Find the fraction.

HINT. Let $\frac{n}{d}$ = the fraction. Then $\frac{n+a}{d}=2$, and $\frac{n}{d+b}=1$.

7. If the numerator of a certain fraction be increased by 1, the value of the resulting fraction is x. If the denominator of the fraction be decreased by 2, the value of the resulting fraction is y. Find the numerator and the denominator.

8. The value of a certain fraction is b. If 2 be added to the numerator, the value of the resulting fraction is c. Find the numerator and the denominator.

9. A boy who weighs a pounds and one who weighs b pounds balance at the opposite ends of a teeter board whose length is l feet. How far is the fulcrum from each boy?

10. A certain number of books at 80 cents each and another number at $1.10 each cost together h dollars. If the prices of the books had been interchanged, the total cost would have been k dollars. Find the number of each kind.

11. Two books cost c dollars. The first cost d cents more than the second. Find the cost of each.

12. A and B have k dollars in all. If A gives h dollars to B they have equal sums. How many dollars had each at first?

13. If A gives h dollars to B, they will have equal sums. If B gives k dollars to A, A will have twice as much as B. How many dollars has each?

14. If A gives $10 to B, B will have h dollars more than A. But if B gives k dollars to A, A will have three times as much as B. How many dollars has each?

15. A and B have together $40. A gives h dollars to B, after which B gives k dollars to A. Then they have equal sums. How many dollars had each at first?

16. A gives r dollars to B and then has $\frac{1}{2}$ as much money as B. Then B gives $8 to A and has left $\frac{3}{4}$ as much money as A. How many dollars had each at first?

17. A part of $1000 is invested at $a\%$ and the remainder at $b\%$. The yearly income from both investments is c dollars. How many dollars are there in each investment?

18. A portion of x dollars is invested at 5% and the remainder at 4%. The yearly income is y dollars. How many dollars are there in each investment?

19. A works three times as fast as B. Together they can do a piece of work in c days. How many days would each require alone?

HINT. Let a and b denote the number of days required by A and B respectively to do the work alone.

Then $$3a = b, \text{ and } \frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

20. A works h times as fast as B. Together they can do a piece of work in 4 days. How many days would each require alone?

21. A and B together can do a piece of work in d days. A can do $\frac{2}{3}$ of the work in 6 days. How many days does each require alone?

22. A and B together can do a piece of work in 5 days. A can do $\frac{2}{5}$ of the work in k days. How many days does each require alone?

23. B requires twice as much time as A to do a piece of work which they can do together in n days. How many days does each require alone?

24. A and B together can do a piece of work in p days. A works q times as fast as B. How many days does each require alone?

100. Linear systems in three unknowns. The method of obtaining the set of roots of a system of linear equations in three unknowns is illustrated in the following:

EXAMPLE

$$\text{Solve the system} \begin{cases} 4x - 2y + z = 9, & (1) \\ 3x + y + 2z = 13, & (2) \\ 2x + 3y - 3z = -2. & (3) \end{cases}$$

Solution. Eliminate one unknown, say z, between (1) and (2), thus:

(1) · 2, $\qquad 8x - 4y + 2z = 18. \qquad$ (4)

(2), $\qquad 3x + y + 2z = 13. \qquad$ (5)

(4) – (5), $\qquad 5x - 5y = 5, \qquad$ (6)

or $\qquad x - y = 1. \qquad$ (7)

Now eliminate z between (1) and (3), as follows:

(1) · 3, $\qquad 12x - 6y + 3z = 27. \qquad$ (8)

(3), $\qquad 2x + 3y - 3z = -2. \qquad$ (9)

(8) + (9), $\qquad 14x - 3y = 25. \qquad$ (10)

Solving (7) and (10) we obtain $\quad x = 2, y = 1.$

Substitute 2 for x and 1 for y in (2),

$$6 + 1 + 2z = 13. \qquad (11)$$

Solving (11), $\qquad z = 3. \qquad$ (12)

Check. Substituting 2 for x, 1 for y, and 3 for z in (1), (2), (3),

$$8 - 2 + 3 = 9, \quad \text{or} \quad 9 = 9.$$

$$6 + 1 + 6 = 13, \quad \text{or} \quad 13 = 13.$$

$$4 + 3 - 9 = -2, \quad \text{or} \quad -2 = -2.$$

The foregoing example illustrates the

Rule. *Decide from an inspection of the coefficients which unknown is most easily eliminated.*

Using any two equations, eliminate that unknown.

With one of the equations just used, and the third equation, again eliminate the same unknown.

The last two operations give two equations in the same two unknowns. Solve these two equations by the rule (pp. 220–221).

Substitute the two values found in the simplest of the original equations and solve for the third unknown.

Check. *Substitute the values found in each of the original equations and simplify results.*

EXERCISES

Solve for the unknowns involved:

1. $$\begin{aligned} x + y + z &= 9, \\ x - y - z &= -3, \\ x + y - z &= 5. \end{aligned}$$

2. $$\begin{aligned} x + y + z &= 3, \\ 2x + y - z &= 6, \\ x - y - 2z &= 0. \end{aligned}$$

3. $$\begin{aligned} x + y + z &= 0, \\ 3x + 2y + 3z &= -1, \\ x - y - 2z &= -8. \end{aligned}$$

4. $$\begin{aligned} 3A + B - C &= -8, \\ A - 4B + 2C &= 9, \\ 2A + 3B + 3C &= 13. \end{aligned}$$

5. $$\begin{aligned} 3x - 2y + 4z &= 9, \\ 2x + 3y - 2z &= -3, \\ 5x + 2y + 3z &= 6. \end{aligned}$$

6. $$\begin{aligned} 6h + 5k + 4m &= 9, \\ 4h + 6k + 5m &= 5, \\ 5h + 4k + 6m &= 16. \end{aligned}$$

7. $$\begin{aligned} x + 3y + 2z &= 17, \\ y - 4z &= -5, \\ x + 2y &= -8. \end{aligned}$$

8. $$\begin{aligned} 3x - y - 2z &= -2, \\ 6x + z &= 4, \\ 3y - 4z &= -11. \end{aligned}$$

9. $$\begin{aligned} x + y &= 1, \\ y + z &= 3, \\ z + x &= 8. \end{aligned}$$

10. $$\begin{aligned} 2p_1 - 3p_2 &= 4, \\ 3p_1 + p_3 &= 5, \\ p_2 - 2p_3 &= 2. \end{aligned}$$

11. $$\begin{aligned} 5x - 7y + 4z &= 3, \\ 6x + 3y - 5z &= -3, \\ 4x + 6y + 3z &= 25. \end{aligned}$$

12. $$\begin{aligned} 3A_1 + 2A_2 + 4A_3 &= 9, \\ A_1 - A_2 - 2A_3 &= 3, \\ A_1 - 3A_2 + 2A_3 &= 2. \end{aligned}$$

13. $$\begin{aligned} \frac{1}{x} + \frac{1}{y} - \frac{1}{z} &= 1, \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= \frac{2}{3}, \\ \frac{1}{x} - \frac{1}{y} + \frac{1}{z} &= 0. \end{aligned}$$

HINT. Solve without clearing of fractions.

14.
$$\begin{gathered}\frac{2}{p}+\frac{10}{q}-\frac{3}{r}=-3,\\ \frac{4}{p}+\frac{5}{q}+\frac{6}{r}=15,\\ \frac{1}{p}+\frac{5}{q}-\frac{1}{r}=-\frac{1}{2}.\end{gathered}$$

15.
$$\begin{gathered}\frac{2}{x}+\frac{3}{z}=\frac{34}{3},\\ \frac{3}{x}+\frac{4}{y}=24,\\ \frac{4}{z}+\frac{5}{y}=28.\end{gathered}$$

PROBLEMS

1. Find three numbers of which the sum of the first and second is 54, the second and third 65, and the first and third 59.

2. The sum of three numbers is 70. The sum and the quotient of two of them are 45 and 5 respectively. Find the numbers.

3. The perimeter of a triangle is 60 feet. Two of its sides are equal, and the third side is 6 feet longer than either of the first two. Find the length of each side.

4. The sum of two sides of a triangle is 53 feet and their difference is 19 feet. The perimeter of the triangle is 62 feet. Find the length of each side.

5. The sum of the two sides of a triangle which meet at one vertex is 41 feet, at another vertex 46 feet, and at the third vertex 57 feet. Find the length of each side.

6. The sum of three numbers is 24. The quotient of two of them is 3, and the sum of these two divided by the third is 5. Find the numbers.

Fact from Geometry. The sum of the three angles of any triangle (each angle being measured in degrees) is 180 degrees.

7. Two of the angles of a triangle are equal, and their sum is equal to the third. Find the number of degrees in each angle.

8. Angle A of a triangle is 12 degrees greater than angle B, and angle B is 21 degrees greater than angle C. How many degrees are there in each?

9. The sum of two angles of a triangle is 30 degrees more than the third, and the third is 15 times the difference of the first two. How many degrees are there in each?

10. A and B together can do a piece of work in 6 days, A and C in 8 days, and B and C in 12 days. Find the time required by each alone and by all together. 276

11. Two pumps together can fill a tank in 6 hours. The first of these and a third together can fill the tank in 8 hours. All three together can fill the tank in 4 hours. Find the number of hours required by each alone. 327

12. The sum of the digits of a three-digit number is 19. The units' digit exceeds the tens' digit by 3. If 495 be added to the number, the result is expressed by the digits in reverse order. Find the number.

13. If the tens' and units' digits of a three-digit number be interchanged, the resulting number is 54 less than the original number. If the tens' and hundreds' digits be interchanged, the resulting number is 360 more than the original number. The sum of the digits is 11. Find the original number.

NOTE. Perhaps the student wonders whether a linear equation in three unknowns has a graphic representation. It may partially satisfy his curiosity to say that by means of three axes at right angles to each other such a representation, though beyond the scope of this book, is possible. Further, the points whose x, y, and z values satisfy the equation lie in a flat surface called a plane. Two such surfaces may intersect in a straight line, and the system of two equations which the surfaces represent is satisfied by the x, y, and z values of any point on this line. If three such surfaces intersect in a single point, the system which the surfaces represent is satisfied by the x, y, and z values of this point. In the systems of equations in three unknowns on page 237 the student is really finding the coördinates of the point of intersection of three planes.

Since space has but three dimensions, this method of representation of linear equations in two or three unknowns cannot be extended to equations containing four or more unknowns.

CHAPTER XIX

SQUARE ROOT

101. Square root of algebraic expressions. The square root of $t^2+2tu+u^2$ is $\pm(t+u)$.

A study of this form will enable us to extract the square root of any polynomial. Obviously the square root of t^2 (the first term of the trinomial) is t, the first term of the root. If t^2 is subtracted from the trinomial, the remainder is $2tu+u^2$. The next term of the root (u) can be found by dividing the first term of the remainder ($2tu$) by $2t$ (twice that term of the root already found).

The work may be arranged thus:

$$\begin{array}{lll|l}
 & & t^2+2tu+u^2 & \underline{t+u} \\
 & & \underline{t^2} & \\
\text{Trial divisor,} & 2t & 2tu+u^2 & \\
\text{Complete divisor,} & 2t+u & \underline{2tu+u^2=(2t+u)u} &
\end{array}$$

Therefore the required roots are $\pm(t+u)$.

The foregoing process is easily extended to extracting the square root of the polynomial $4x^4-20x^3+37x^2-30x+9$, whose square root contains *three* terms, as follows:

$$\begin{array}{ll|l}
 & 4x^4-20x^3+37x^2-30x+9 & \underline{2x^2-5x+3} \\
(2x^2)^2= & \underline{4x^4} & \\
\text{First trial divisor, } 2\cdot 2x^2=4x^2 & -20x^3+37x^2 & \\
\text{First complete divisor, } 4x^2-5x & \underline{-20x^3+25x^2=(4x^2-5x)(-5x)} & \\
\text{Second trial divisor,} & & \\
\quad 2(2x^2-5x)=4x^2-10x & 12x^2-30x+9 & \\
\text{Second complete divisor, } 4x^2-10x+3 & \underline{12x^2-30x+9=(4x^2-10x+3)3} &
\end{array}$$

Therefore the required roots are $\pm(2x^2 - 5x + 3)$.

The term $2x^2$ was obtained by taking the square root of $4x^4$; the second term, $-5x$, by dividing $-20x^3$ by the first trial divisor, $4x^2$; and the third term, 3, by dividing $12x^2$ by $4x^2$, the first term of the second trial divisor.

The method just illustrated may be stated in the

Rule. *Arrange the terms of the polynomial according to descending powers of some letter in it.*

Extract the square root of the first term. Write the result (with plus sign only) as the first term of the root and subtract its square from the given polynomial.

Double the root already found for the first trial divisor, divide the first term of the remainder by it, and write the quotient as the second term of the root.

Annex the quotient just found to the trial divisor, making the complete divisor; multiply the complete divisor by the second term of the root and subtract the product from the last remainder.

If terms of the polynomial still remain, double the root already found for a trial divisor, divide the first term of the trial divisor into the first term of the remainder, write the quotient as the next term of the root, form the complete divisor, and proceed as before until the process ends, or until the required number of terms of the root have been found.

Inclose the root thus found in a parenthesis preceded by the sign ±.

NOTE. The process of extracting the square root of numbers was familiar to mathematicians long before they knew how to find the square root of polynomials. This is consistent with the fact that the development of the methods of performing operations on literal number symbols generally followed and grew out of the similar operations on numerals. The application of the rules for extracting the square root of numbers to that of polynomials is generally ascribed to Recorde (1510–1558), who was the author of the earliest English work on algebra that we know. This book, which bears the title "The Whetstone of Wit," gives an accurate idea of the algebraic knowledge of the time, and had a very wide influence.

EXERCISES

Obtain one square root of:

1. $a^2 + 6a + 9.$
2. $x^2 + 10ax + 25a^2.$
3. $x^4 + 16 - 8x^2.$
4. $4a^2x^2 + 4x^4 + a^4.$
5. $a^4 + 3a^2 + 2a^3 + 2a + 1.$
6. $24x^2 - 32x + 16 + x^4 - 8x^3.$
7. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$
8. $21c^2 + c^4 + 20c - 10c^3 + 4.$
9. $n^6 + 9n^2 + 10n^3 + 25 - 6n^4 - 30n.$
10. $c^4 - 12c + 9c^2 + 4 + 4c^2 - 6c^3.$
11. $5a^4 + 12a^5 + 16 - 23a^2 + 4a^6 + 8a - 22a^3.$
12. $c^4 - 4c^3a + 6c^2a^2 - 4ca^3 + a^4.$
13. $30xy^3 + 25y^4 - 11x^2y^2 - 12x^3y + 4x^4.$
14. $-36a^4x + 36a^2x^2 + 9a^6 - 24a^3x^2 + 16x^4 + 48ax^3.$
15. $9c^4 - 2a^2b^2c^2 + 4a^3b^3c + a^4b^4 - 12abc^3.$
16. $2a^2xc^3 - 4xc^3 - 4a^2x^2 + 4x^2 + c^6 + a^4x^2.$
17. $4 - \frac{4c^2}{5} + \frac{c^4}{25}.$
18. $\frac{9}{a^2} + \frac{a^2}{4} - 3.$
19. $x^4 - 4x^3 + 5x^2 - 2x + \frac{1}{4}.$
20. $a^4 + 4a^3 + 4a^2 - \frac{4a^2}{3} - \frac{8a}{3} + \frac{4}{9}.$
21. $x^4 - x^3 + \frac{x^2}{4} + \frac{2x^2}{3} - \frac{x}{3} + \frac{1}{9}.$
22. $\frac{25m^4}{4} - \frac{127m^2}{18} - 2m + \frac{10m^3}{3} + \frac{9}{4}.$

102. Square root of arithmetical numbers. Since $1 = 1^2$, and $81 = 9^2$, a one-digit or a two-digit square has only *one* digit in its square root.

And as $100 = 10^2$, and $9801 = (99)^2$, a three-digit or a four-digit square has *two* digits in its square root.

Also $10,000 = 100^2$, and $998,001 = (999)^2$; hence a five-digit or a six-digit square has *three* digits in its square root.

These examples illustrate the relation between the number of digits in a number and the number of digits in its square root. They also suggest a method of obtaining the first digit in the square root of any number.

For example, take the four numbers 78′43′56, 7′84′35, .98′01, and .03′27′4. Beginning at the decimal point in each, point off periods of two digits each, as indicated. Any period incomplete on the right, as in .03′27′4, should be completed by annexing one zero; thus, .03′27′40. Now the first digit in the square root is the greatest integer whose square is less than or equal to the left-hand period. This is true whether the latter contains *two* digits or *one*. Hence the first digit in the square root of 78′43′56 is 8, in the square root of 7′84′35 is 2, in the square root of .98′01 is 9, and in the square root of .03′27′40 is 1.

Moreover, the number of digits in the square root of a perfect square is equal to the number of periods, provided a *single digit* remaining on the left is called a period.

Just how t and u are involved in the square of $(t + u)$, or $t^2 + 2\,tu + u^2$, is obvious on inspection, because the parts t^2, $2\,tu$, and u^2 cannot be united into one term. In the square of an arithmetical number, however, the parts are united. Thus $(53)^2 = (50 + 3)^2 = 2500 + 300 + 9 = 2809$. Now it is clear how 50 and 3 are involved in $2500 + 300 + 9$, but it is not plain from 2809 alone. Pointing off, however, enables us to discover at once the first digit, 5, which is equivalent to 5 tens, or 50. With the exception of pointing off, the method of extracting the square root of an arithmetical number does not differ greatly from the method of extracting the square root of an algebraic expression. In fact, the formula, the square root of $t^2+2tu+u^2=\pm(t+u)$, can be used to explain the two processes.

If t denotes the tens and u the units, $t^2+2tu+u^2$ is closely related to $2500+300+9$, t^2 being 2500, or $(50)^2$; u^2 being 9, or 3^2; and $2tu$ being $2\cdot 50\cdot 3$. Therefore the process of extracting the square root of 2809 may be based on these relations and the work arranged as follows:

$$\begin{array}{r|l}
 & 2809\,\underline{|50+3} \\
t^2=(50)^2 & 2500 \\
2t=2\cdot 50=100 & 309 \\
2t+u=100+3 & 309=(100+3)3=(2t+u)u=2tu+u^2 \\
\hline
\end{array}$$

Therefore ± 53 are the two square roots of 2809.

If the number has three digits in its square root, the work and explanations may be arranged thus:

$$\begin{array}{l|l}
 & 1'74'24\,\underline{|100+30+2} \\
t^2=(100)^2 & 1\,00\,00=10 \text{ tens squared} \\
\text{First trial divisor,} & \\
\quad 2t=2\cdot 100=200 & 74\,24 \\
\text{First complete divisor,} & \\
2t+u=200+30=230 & 69\,00=(2\cdot 10 \text{ tens}+30 \text{ units})\,30 \\
\text{Second trial divisor,} & \\
\quad 2t=2\cdot 130=260 & 5\,24 \\
\text{Second complete divisor,} & \\
2t+u=260+2=262 & 5\,24=(2\cdot 13 \text{ tens}+2 \text{ units})\,2 \\
\end{array}$$

Therefore ± 132 are the square roots of 17,424.

When the method and reasons for the process have become familiar, the work may be shortened by omitting the explanations and unnecessary zeros as follows:

$$
\begin{array}{r|l}
\multicolumn{2}{l}{\quad 28'09\,\underline{|\,53}} \\
\multicolumn{2}{l}{\quad 25} \\ \cline{2-2}
103 & 309 \\
 & 309 \\ \cline{2-2}
\end{array}
\qquad\qquad
\begin{array}{r|l|l}
\multicolumn{3}{l}{\quad 1'74'24\,\underline{|\,132}} \\
\multicolumn{3}{l}{\quad 1} \\ \cline{2-3}
23 & \multicolumn{2}{l}{74} \\
 & \multicolumn{2}{l}{69} \\ \cline{2-3}
\multicolumn{2}{r|}{262} & 524 \\
\multicolumn{2}{r|}{} & 524 \\ \cline{3-3}
\end{array}
$$

The method of the two preceding solutions is the one commonly used for extracting the positive square root of a number. For it we have the

Rule. *Begin at the decimal point and point off as many periods of two digits each as possible: to the left if the number is an integer; to the right if it is a decimal; to both the left and the right if the number is part integral and part decimal.*

Find the greatest integer whose square is equal to or less than the left-hand period, and write this integer for the first digit of the root.

Square the first digit of the root, subtract its square from the first period, and annex the second period to the remainder.

Double the part of the root already found for a trial divisor, divide it into the remainder (omitting from the latter the right-hand digit), and write the integral part of the quotient as the next digit of the root.

Annex the root digit just found to the trial divisor to make the complete divisor, multiply the complete divisor by this root digit, subtract the result from the dividend, and annex to the remainder the next period for a new dividend.

Double the part of the root already found for a new trial divisor and proceed as before until the desired number of digits of the root have been found.

After extracting the square root of a number involving decimals, point off one decimal place in the root for every decimal period in the number.

Check. *If the root is exact, square it. The result should be the original number. If the root is inexact, square it and add to this result the remainder. The sum should be the original number.*

Sometimes in using a trial divisor we obtain too great a quotient for the next digit of the root. This happens in obtaining the second digit of the square root of 32,301, where 2 into 22 gives 11. Obviously 10 and 11 are both impossible. If 9 is tried we get $9 \cdot 29$, or 261, which is greater than 223. Similarly, 8 is too great. But $7 \cdot 27 = 189$, which is less than 223. Therefore 7 is the second digit of the root.

$$\begin{array}{r|l}
 & 3'23'01 \;\underline{|\,1} \\
 & \underline{1} \\
2 & 223
\end{array}$$

Occasionally the trial divisor gives a quotient less than 1. This indicates that the required root digit is 0, which should be written in the root and the work continued as usual. An instance of this kind occurs in finding the second digit in the square root of 9′42.49. The quotient of $4 \div 6$ is $\frac{2}{3}$, which is not an integer. Therefore the second digit of the root is 0. Then the next period, 49, should be brought down. The new trial divisor will be 60, which will give 7 as the third digit of the root. The work can easily be completed, giving 30.7 as the square root.

$$\begin{array}{r|l}
 & 9'42.49 \;\underline{|\,30} \\
 & \underline{9} \\
6 & 42
\end{array}$$

An attempt to extract the square root of 3 by annexing decimal periods of zeros and applying the rule becomes a never-ending process.

The number 3 has no exact square root, and no matter how far the work is carried, there is no final digit. As the work stands, we know that the square root of 3 lies between 1.732 and 1.733.

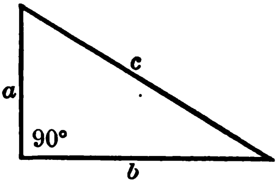
$$\begin{array}{r|l}
 & 3.00'00'00 \;\underline{|\,1.732} \\
 & \underline{1} \\
27 & 200 \\
 & \underline{189} \\
343 & 1100 \\
 & \underline{1029} \\
3462 & 7100 \\
 & \underline{6924}
\end{array}$$

EXERCISES

Obtain the positive square root, to three decimal places, of:

1. 4489.
2. 5184.
3. 5329.
4. 6241.
5. 9216.
6. 16129.
7. 24649.
8. 43436.
9. 53361.
10. 165680.
11. 223729.
12. 328329.

Fact from Geometry. In the adjacent right triangle, $a^2 + b^2 = c^2$; the sides a and b, which form the right angle, are called the **legs**; and c, the side opposite the right angle, is called the **hypotenuse.**



If leg a is 8 and leg b is 15, then substituting in $a^2 + b^2 = c^2$ gives $64 + 225 = c^2$. Whence $289 = c^2$ and $c = \pm 17$.

Since -17 is not a practical answer, it is rejected.

In Exercises 13–16 find the hypotenuse and the area of a right triangle whose legs are:

13. 63 and 16.
14. 48 and 55.
15. 104 and 153.
16. 645 and 812.

In Exercises 17–19 find the other leg and the area of a right triangle in which the hypotenuse and one leg are respectively:

17. 109 and 91. **18.** 2.57 and .32. **19.** 2.05 and 1.87.

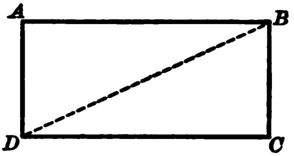
Extract the square root in Exercises 20–23 inclusive to four decimals, and in Exercises 24–28 inclusive to three decimals.

A common fraction or the fractional part of a mixed number should be reduced to a decimal before extracting the square root, unless the root is seen to be exact.

20. 6.4271.
21. 884.3.
22. .0869.
23. .00321.
24. 3.
25. 6.
26. $2\frac{3}{5}$.
27. $2\frac{3}{7}$.
28. $\frac{3}{17}$.

(In the following find all inexact answers to two decimals.)

In rectangle $ABCD$ line DB is called a **diagonal.**



29. Find the diagonal of a rectangle whose adjacent sides are 28 feet and 195 feet respectively.

30. One diagonal of a rectangle is 409 and one side is 391. Find the other side and the area.

31. One diagonal of a rectangle is 533 and one side is 92. Find the perimeter of the rectangle.

32. A rectangle is 7 yards longer than it is wide. Its perimeter is 138 feet. Find one diagonal.

33. One diagonal of a square is 74 meters. Find the side.

34. The side of a square is 52 inches. Find one diagonal.

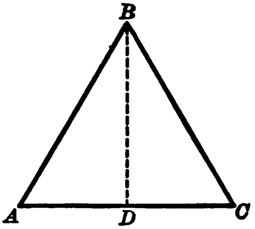
35. One leg of a right triangle is 10. The hypotenuse is twice the other leg. Find the hypotenuse and the other leg.

36. The hypotenuse of a right triangle is three times one leg and the other leg is 16. Find the sides.

37. A rectangle is 2.4 times as long as it is wide. One diagonal is 52. Find the length and the width.

38. The width of a rectangle is 25% less than the length. The diagonal is 100. Find the area.

39. The length of a rectangle is 10. The diagonal is twice the shorter side. Find the width.



Fact from Geometry. A line drawn from one vertex of an equilateral triangle to the middle point of the opposite side is perpendicular to that side.

Then, in the equilateral triangle ABC, if D is the middle point of AC, BD is the altitude; and

$$\overline{BD}^2 = \overline{AB}^2 - \overline{AD}^2 = \overline{AB}^2 - \left(\frac{AC}{2}\right)^2.$$

40. If BC in the foregoing triangle is 6, find BD and the area of the triangle.

41. If AC in the foregoing triangle is 8, find BD and the area of the triangle.

42. If BD in the foregoing triangle is 14, find AB and the area of the triangle.

43. The perimeter of an equilateral triangle is 45. Find the altitude.

44. The altitude of an equilateral triangle is 25 centimeters. Find one side.

NOTE. A method of extracting the square root of numbers not unlike that in use to-day was employed by the Greek, Theon, about A.D. 350. In the Middle Ages square roots were extracted with a fair degree of accuracy by using the formulas of approximation:

$$(1)\ \sqrt{a^2+x} = a + \frac{x}{2a}. \qquad (2)\ \sqrt{a^2+x} = a + \frac{x}{2a+1}.$$

The true value of the square root of the number was proved to be between the results obtained by these expressions. Thus, if $\sqrt{65}$ was desired, it was noticed that $65 = 64 + 1$, and from (1)

$$\sqrt{65} = \sqrt{64+1} = \sqrt{8^2+1} = 8 + \frac{1}{2 \cdot 8} = 8\tfrac{1}{16},$$

while from (2)

$$\sqrt{65} = \sqrt{64+1} = \sqrt{8^2+1} = 8 + \frac{1}{2 \cdot 8 + 1} = 8\tfrac{1}{17}.$$

Thus the true value of $\sqrt{65}$ is between these two numbers. This method was known to the Arabs.

It should be kept in mind that the use of decimal fractions and of the decimal point was not common until the eighteenth century. Consequently the complete development of the method of extracting the square root given in the text is comparatively recent.

Simplify [illegible]

[illegible] $= (4a^2b^4)$ [illegible]

$\sqrt{18} = 3/6\sqrt{2}$

$\sqrt[6]{9} = \sqrt[6]{3^2} = 3^{1/3} = \sqrt[3]{3}$

CHAPTER XX

RADICALS

103. Rational numbers. The quotient of two integers is called a **rational number.**

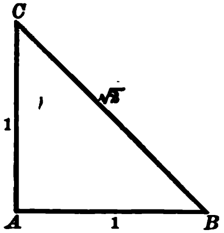
Any integer is a rational number, since it may be considered as the quotient of itself and 1.

Thus 5, -8, $\frac{2}{3}$, $-\frac{7}{8}$, and 4.693 are rational numbers.

104. Radical. A radical is an indicated root of an algebraic or arithmetical expression.

Thus $\sqrt{4}$, $\sqrt{3}$, $\sqrt[3]{a}$, and $\sqrt{x^2 - 5x + 6}$ are radicals.

If a number under a radical sign is such that the root cannot be taken exactly, the radical represents an **irrational number.**



Thus $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt{3}$, $\sqrt[3]{5}$, are irrational numbers, since the indicated roots of 2, of 3, and of 5 will never come out even however far the process of extracting the root is carried.

Though no irrational numbers can be expressed exactly in decimals, we can represent a few of them by the lengths of lines. Thus in the right triangle ABC, if $AB = AC = 1$ inch, $BC = \sqrt{2}$ inches. If AB were 2 inches and AC were 1 inch, BC would be $\sqrt{5}$ inches.

There are other types of irrational numbers which cannot be expressed in terms of radicals, but their consideration is too complicated for this text.

If a negative number occurs under a square-root sign, the radical represents an imaginary number.

$2c^2(20 - i)$

$\frac{1}{2} i^2$

Thus $\sqrt{-2}$, $\sqrt{-8}$, and $\sqrt{-4}$ are imaginary numbers.

If the student pursues the study of algebra he will learn that imaginary numbers are required to express completely the cube and higher roots of any positive or negative number.

For example, he will learn that the number 27 has *two other* cube roots besides the number 3.

105. Index. The small figure like the 4 in $\sqrt[4]{\ }$ is called the index of the radical.

The index determines the **order** of the radical and indicates the root to be extracted.

In $5\sqrt[3]{7}$, 3 is the index, and the radical is of the third order.

106. Radicand. The **radicand** is the number, or expression, under the radical sign.

In $\sqrt{7}$ and $\sqrt[3]{ax}$, 7 and ax are the radicands.

107. Principal root. For a given index the **principal root** of a number is its *one root*, if it has but one; or its *positive root*, if it has two roots.

The principal root of $\sqrt[3]{-27}$ is -3; that of $\sqrt[4]{16}$ is $+2$, not -2.

108. Fractional exponents. Radical expressions may be written in two ways, with *radical signs* or with *fractional exponents*. The relation between the two will now be explained. To do this it is necessary to extend the meaning of the term *exponent*, which as defined on page 9 applied to positive integral exponents only. We shall assume that the laws which govern the use of integral exponents hold for fractional exponents also.

The fact that $x^2 \cdot x^3 = x^5$ illustrates the more general law $x^a \cdot x^b = x^{a+b}$, where a and b represent either integers or fractions.

Accordingly $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1$ or x. Since $x^{\frac{1}{2}}$ multiplied by itself gives x, $x^{\frac{1}{2}}$ *must be another way of writing the square root of* x.

Hence $\sqrt{x}$ may be written $x^{\frac{1}{2}}$.

Then $4^{\frac{1}{2}} = \sqrt{4} = 2$, and $(25\,a^2)^{\frac{1}{2}} = \sqrt{25\,a^2} = 5\,a$.

Further, $x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x^1 = x$.

And since $x^{\frac{1}{3}}$ is one of the three equal numbers whose product is x, $x^{\frac{1}{3}}$ *is another way of writing the cube root of* x

Therefore $\sqrt[3]{x}$ may be written $x^{\frac{1}{3}}$.

This means that $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$.

Similarly, $\sqrt[4]{x} = x^{\frac{1}{4}}$.

In general terms, $\sqrt[n]{x} = x^{\frac{1}{n}}$.

Now $x^{\frac{3}{2}} = x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = (x^{\frac{1}{2}})^3 = (\sqrt{x})^3$,

and $x^{\frac{3}{2}} = x^{3 \cdot \frac{1}{2}} = (x^3)^{\frac{1}{2}} = \sqrt{x^3}$.

Hence $x^{\frac{3}{2}} = (\sqrt{x})^3$, or $\sqrt{x^3}$.

In general terms, $x^{\frac{a}{n}} = \sqrt[n]{x^a}$.

Thus $x^{\frac{a}{n}}$ means the nth root of x to the ath power.

The student should fix in mind that the *denominator* of the fractional exponent is the *index* of the root, and the *numerator* the *power* to which the radicand is raised. Moreover, whether one extracts the root first and then raises the result to the power, or vice versa, depends wholly on convenience.

ORAL EXERCISES

Read in radical form:

1. $x^{\frac{2}{3}}$.
2. $x^{\frac{3}{2}}$.
3. $(cd)^{\frac{3}{2}}$.
4. $(2\,x)^{\frac{1}{3}}$.
5. $2\,x^{\frac{1}{3}}$.
6. $5\,a^{\frac{3}{4}}$.
7. $(5\,a)^{\frac{3}{4}}$.
8. $3\,rx^{\frac{2}{3}}$.
9. $h^{\frac{2}{3}}k^{\frac{2}{3}}$.
10. $7\,s^{\frac{2}{3}}(t+w)^{\frac{1}{2}}$.
11. $3\,a^{\frac{5}{2}}(bc)^{\frac{1}{3}}$.
12. $5^{\frac{1}{2}}x^{\frac{2}{3}}$.
13. $4^{\frac{1}{3}}t^{\frac{1}{a}}$.
14. $2\,x^{\frac{1}{b}}y^{\frac{c}{d}}$.

Find the numerical values of:

15. $25^{\frac{1}{2}}$.

16. $27^{\frac{1}{3}}$.

17. $16^{\frac{1}{4}}$.

18. $4^{\frac{3}{2}}$.

19. $64^{\frac{2}{3}}$.

20. $125^{\frac{2}{3}}$.

21. $(-8)^{\frac{4}{3}}$.

22. $32^{\frac{1}{5}}$.

23. $81^{\frac{3}{4}}$.

24. $(-216)^{\frac{2}{3}}$.

25. $(\frac{1}{16})^{\frac{1}{2}}$.

26. $(\frac{1}{25})^{\frac{1}{2}}$.

27. $(\frac{1}{16})^{\frac{1}{4}}$.

28. $25^{\frac{1}{2}} \cdot 4^{\frac{5}{2}}$.

29. $4^{\frac{1}{2}} \cdot (\frac{1}{4})^{\frac{1}{2}}$.

30. What is the principal square root of 4? of 25? of 36? the principal cube root of $+8$? of -8? of -27? of $+27$? the principal fifth root of 32? of -32? of 243? of -243?

31. What is an index? a radicand? Illustrate.

EXERCISES

Write with fractional exponents and simplify results:

1. $\sqrt{a^3}$.

2. $\sqrt{ax^4}$.

3. $3\sqrt{2x^5}$.

4. $\sqrt{9x}$.

5. $5\sqrt{16ax^2}$.

6. $\sqrt[3]{a^2}$.

7. $\sqrt[3]{ax^4}$.

8. $2\sqrt[3]{2x^2}$.

9. $3\sqrt[3]{8x^4}$.

10. $4\sqrt[3]{27ax^3}$.

11. $2\sqrt[4]{a^2x^3}$.

12. $4\sqrt[4]{16x}$.

13. $12x^2\sqrt[3]{ax^8}$.

14. $c\sqrt{(de)^3}$.

15. $3\sqrt{a^3} \cdot \sqrt[3]{x^2}$.

16. $\sqrt[n]{x^a} \cdot \sqrt[n]{y^b}$.

17. $\sqrt[n]{x^a} \cdot \sqrt[n]{x^{3a}}$.

18. $(-32)^{\frac{4}{5}} \cdot \sqrt[3]{-64}$.

19. $36^{\frac{1}{2}} \cdot \sqrt{\frac{1}{9}}$.

20. $9^{\frac{1}{2}} \cdot \sqrt[3]{27}$.

21. $(\frac{1}{4})^{\frac{1}{2}} \cdot \sqrt[3]{8^2}$.

109. Simplification of radicals. The form of a radical expression may be changed without altering its numerical value. Such changes are necessary for many reasons. For example, the numerical value of a radical expression is most easily obtained from its simplest form. It will be made clear later that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Granting that the two fractions are really equal, one can see by inspection that the value to several decimals can be computed more easily from the second fraction than from the first.

EXAMPLES

Study the following changes of form:

1. $\sqrt{36} = \sqrt{4 \cdot 9} = \sqrt{4} \cdot \sqrt{9} = 2 \cdot 3 = 6.$
2. Similarly, $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}.$
3. More generally, $\sqrt{a^3b} = \sqrt{a^2}\sqrt{ab} = a\sqrt{ab}.$
4. Also $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = \sqrt[3]{8}\sqrt[3]{3} = 2\sqrt[3]{3}.$
5. More generally, $\sqrt[3]{a^3b} = \sqrt[3]{a^3}\sqrt[3]{b} = a\sqrt[3]{b}.$
6. Finally, $\sqrt[n]{a^nb} = \sqrt[n]{a^n}\sqrt[n]{b} = a\sqrt[n]{b}.$

The six preceding examples illustrate the correctness of the following rule for simplifying a radical involving the square root of an integer or an integral expression.

Rule. *Separate the radicand into two factors one of which is the greatest perfect square which it contains. Then take the square root of this factor and write it as the coefficient of a radical of which the other factor is the radicand.*

If the radical already has a coefficient other than the number 1, multiply the result obtained above by this coefficient.

A similar rule holds for radicals involving the cube and higher roots.

Thus $\sqrt[3]{16} = \sqrt[3]{8 \cdot 3} = 2\sqrt[3]{3},$

and $\sqrt[5]{96} = \sqrt[5]{32 \cdot 3} = 2\sqrt[5]{3}.$

NOTE. Although the Arabs were by no means able to state all the rules explained in this chapter, it is interesting to note that they did recognize the truth of a few of them. For instance, a writer about A.D. 830 gives, in his own notation, of course, the facts contained in the formulas $\sqrt{a^2b} = a\sqrt{b}$, and $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

EXERCISES

Simplify:

1. $\sqrt{8}$.
2. $\sqrt{12}$.
3. $\sqrt{18}$.
4. $\sqrt{20}$.
5. $\sqrt{45}$.
6. $\sqrt{50}$.
7. $\sqrt{75}$.
8. $\sqrt{63}$.
9. $\sqrt{98}$.
10. $2\sqrt{72}$.
11. $3\sqrt{80}$.
12. $5\sqrt{128}$.
13. $\sqrt{147}$.
14. $3\sqrt{162}$.
15. $10\sqrt{175}$.
16. $\sqrt{192}$.
17. $\sqrt{243}$.
18. $\sqrt{245}$.
19. $\sqrt{343}$.
20. $\sqrt{363}$.
21. $2\sqrt{720}$.
22. $\sqrt{1250}$.
23. $\sqrt[3]{16}$.
24. $\sqrt[3]{24}$.
25. $2\sqrt[3]{40}$.
26. $3\sqrt[3]{56}$.
27. $4\sqrt[3]{72}$.
28. $5\sqrt[3]{96}$.
29. $\sqrt[3]{54}$.
30. $\sqrt[3]{81}$.
31. $\sqrt[3]{135}$.
32. $3\sqrt[3]{189}$.
33. $7\sqrt[3]{128}$.
34. $\sqrt[3]{192}$.
35. $\sqrt[3]{250}$.
36. $8\sqrt[3]{375}$.
37. $\sqrt[3]{448}$.
38. $2\sqrt[3]{625}$.
39. $\sqrt{a^3}$.

HINT. $\sqrt{a^3} = \sqrt{a^2 \cdot a}$.

40. $\sqrt{a^5}$.
41. $\sqrt{4x^3}$.
42. $\sqrt{8x^2}$.
43. $\sqrt{8x^3}$.
44. $\sqrt{a^3x^2}$.
45. $2\sqrt{x^3a}$.
46. $3\sqrt{8a^3x}$.
47. $5x\sqrt{4a^5x^3}$.
48. $\sqrt[3]{a^4}$.

HINT. $\sqrt[3]{a^4} = \sqrt[3]{a^3 \cdot a}$.

49. $\sqrt[3]{a^5}$.
50. $2a\sqrt[3]{x^7}$.
51. $\sqrt[3]{8x^4}$.
52. $\sqrt[3]{27x^5}$.
53. $3x\sqrt[3]{4a^3x}$.
54. $5x\sqrt[3]{4a^2x^3}$.
55. $\sqrt[3]{27a^3x}$.
56. $\sqrt[3]{56a^4x}$.

57. $\sqrt{8+4\sqrt{2}}$.

Solution. $\sqrt{8+4\sqrt{2}} =$
$\sqrt{4(2+\sqrt{2})} =$
$2\sqrt{2+\sqrt{2}}$.

58. $\sqrt{4-4\sqrt{3}}$.
59. $\sqrt{18+9\sqrt{3}}$.
60. $\sqrt{100-25\sqrt{5}}$.
61. $\sqrt{9\sqrt{2}-27}$.
62. $\sqrt{R^2-R^2\sqrt{3}}$.
63. $\sqrt{R^2+4R^2\sqrt{5}}$.
64. $\sqrt{5^2\sqrt{3}-25^2}$.
65. $\sqrt{a^2x^2+a^3\sqrt{x}}$.

The foregoing exercises are easier to simplify than radicals whose radicands are fractions or fractional expressions. The latter arise frequently in practice, especially in certain parts of geometry.

W-J /

EXAMPLES

Study the following simplification of fractional radicands:

1. $\sqrt{\frac{3}{2}} = \sqrt{\frac{6}{4}} = \sqrt{\frac{1}{4}\cdot 6} = \sqrt{\frac{1}{4}}\sqrt{6} = \frac{1}{2}\sqrt{6}.$

2. $6\sqrt{\frac{1}{3}} = 6\sqrt{\frac{3}{9}} = 6\sqrt{\frac{1}{9}\cdot 3} = 6\cdot\frac{1}{3}\sqrt{3} = 2\sqrt{3}.$

3. $\sqrt{\frac{3}{5x}} = \sqrt{\frac{15x}{25x^2}} = \sqrt{\frac{1}{25x^2}\cdot 15x} = \frac{1}{5x}\sqrt{15x}.$

These examples illustrate the following rule for simplifying a square root which has a *fractional* radicand.

Rule. *Multiply the numerator and the denominator of the radicand by the least whole number or simplest expression which will make the resulting denominator a perfect square.*

Then separate the radicand into two factors, one of which is a fraction and at the same time the greatest perfect square which the radicand contains.

Take the square root of this factor and write it as the coefficient of the radical of which the other factor is the radicand. If the original radical has a coefficient, multiply the result as obtained above by this cofficient.

A similar rule holds for simplifying a cube root and radicals of higher orders which have fractional radicands.

EXERCISES

Simplify the following

1. $\sqrt{\frac{3}{4}}$.
2. $\sqrt{\frac{5}{9}}$.
3. $\sqrt{\frac{3}{16}}$.
4. $\sqrt{\frac{1}{2}}$.
5. $\sqrt{\frac{2}{3}}$.
6. $\sqrt{\frac{3}{2}}$.
7. $\sqrt{\frac{3}{5}}$.
8. $\sqrt{\frac{5}{3}}$.
9. $\sqrt{\frac{4}{7}}$.
10. $\sqrt{\frac{7}{5}}$.
11. $\sqrt{\frac{5}{8}}$.
12. $\sqrt{\frac{5}{2}}$.
13. $3\sqrt{\frac{3}{7}}$.
14. $2\sqrt{\frac{2}{5}}$.
15. $\sqrt{\frac{8}{5}}$.
16. $\sqrt{\frac{5}{12}}$.
17. $3\sqrt{\frac{7}{12}}$.
18. $2\sqrt{\frac{5}{18}}$.
19. $3a\sqrt{\frac{3}{11}}$.
20. $3a^2\sqrt{\frac{3}{32}}$.

RADICALS

21. $\sqrt{1-(\frac{1}{2})^2}$.

Hint. $\sqrt{1-(\frac{1}{2})^2} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}}$, etc.

~~**22.**~~ $\sqrt{1-(\frac{1}{3})^2}$.

~~**23.**~~ $\sqrt{9-(\frac{3}{2})^2}$.

~~**24.**~~ $\sqrt{25-(\frac{5}{2})^2}$.

~~**25.**~~ $\sqrt{2+(\frac{3}{8})^2}$.

~~**26.**~~ $\sqrt{81-(\frac{9}{2})^2}$.

27. $\sqrt{121-(\frac{11}{2})^2}$.

28. $\sqrt{169-(\frac{13}{2})^2}$.

29. $\sqrt{\frac{R}{3}}$.

30. $\sqrt{\frac{R^2}{3}}$.

~~**31.**~~ $\sqrt{\frac{3x^2}{4}}$.

32. $\sqrt{R^2+R^2}$.

33. $\sqrt{R^2-\left(\frac{R}{2}\right)^2}$.

~~**34.**~~ $\sqrt{R^2-\left(\frac{R}{4}\right)^2}$.

35. $\sqrt{\frac{3R^2}{4}-\frac{R^2}{4}\sqrt{2}}$.

~~**36.**~~ $\sqrt{\frac{R^2}{4}+\frac{R^2x}{2}}$.

~~**37.**~~ $\sqrt{\frac{R^2}{4}+\frac{R^2}{2}\sqrt{2}}$.

~~**38.**~~ $\sqrt{R^2\sqrt{2}-\frac{R^2}{2}}$.

39. $\sqrt[3]{R^3-\left(\frac{R}{2}\right)^3}$.

~~**40.**~~ $\sqrt[3]{16+8\sqrt{2}}$.

41. $\sqrt[3]{81+3\sqrt{243}}$.

BROWN

YALE

The need for simplifying radicals presents itself in various problems, as, for example, in simple geometrical work on right triangles.

PROBLEMS

(Obtain answers in *simplest radical form.*)

1. One leg of a right triangle is 8 and the other is 10. Find the hypotenuse.

Solution. $x = \sqrt{8^2+10^2} = \sqrt{164} = \sqrt{4 \cdot 41} = 2\sqrt{41}$.

2. The hypotenuse of a right triangle is 8 and one leg is 4. Find the other leg and the area.

3. The hypotenuse of a right triangle is R and one leg is $\frac{R}{2}$. Find the other leg and the area.

4. Find the diagonal of a square whose side is 12.

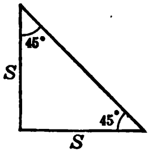
5. Find the sides of a square whose diagonal is 12.

6. Find the sides of a square whose diagonal is $2R$.

Problems involving the following classes of triangles are of frequent occurrence in practical work and often require the use of radicals:

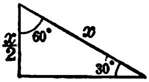
(*a*) An isosceles right triangle; that is, a right triangle with two equal sides.

As indicated in the figure, if each leg is S the two acute angles are 45° each.



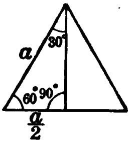
(*b*) A right triangle with one angle 30° or 60°.

As indicated in the figure, if one acute angle is 30° the other is 60°, and vice versa. More important still, the *hypotenuse* is always *twice the shorter leg*.



(*c*) An equilateral triangle.

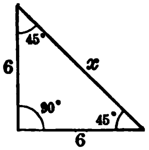
As indicated in the figure, when the altitude is drawn, it divides the base into two equal parts, and in each of the right triangles formed the same relations exist as in (*b*), above.



7. If each leg of an isosceles right triangle is 6, find the hypotenuse.

HINT. $x^2 = 6^2 + 6^2$.

$x = \sqrt{36 + 36}$, etc.



8. Find the hypotenuse of an isosceles right triangle if one leg is 8; if one leg is 13.

9. The hypotenuse of an isosceles right triangle is 10. Find the legs.

HINT. $x^2 + x^2 = 100$. $x^2 = 50$, etc.

10. The hypotenuse of an isosceles right triangle is 13. Find the legs.

11. One angle of a right triangle is 30°. The hypotenuse is 20. Find the other two sides.

HINT. See (*b*), above.

12. One angle of a right triangle is 60°. The hypotenuse is 12. Find the other two sides.

13. One angle of a right triangle is 30°. The leg opposite it is 10. Find the hypotenuse and the other leg.

14. One angle of a right triangle is 60° and the adjacent leg is 12. Find the other two sides.

15. The side of an equilateral triangle is 12. Find the altitude and the area.

Hint. See (*c*) and (*b*), page 258.

16. The side of an equilateral triangle is S. Find the altitude and the area.

17. The altitude of an equilateral triangle is 10. Find the side and the area.

18. The legs of a right triangle are equal. Its hypotenuse is 20. Find the legs of the triangle.

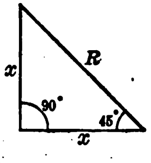
19. The legs of a right triangle are equal and its area is 50. Find the hypotenuse.

20. The legs of a right triangle are $\frac{R}{2}$ and $\frac{3R}{2}$. Find the hypotenuse.

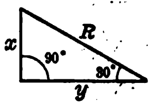
21. One leg of a right triangle is $\frac{R}{5}$. The hypotenuse is R. Find the other leg.

Find x and y in terms of R in the following:

22.



23.



24. The legs of a right triangle are R and $\frac{R}{2}(a-1)$. Find the hypotenuse.

110. Addition and subtraction of radicals. Radicals of the same order which are in their simplest form and have like radicands are really similar terms. They can be added or subtracted and the result expressed by one term according to the rule on page 33.

Thus $3\sqrt{2}-7\sqrt{2}+9\sqrt{2}=5\sqrt{2}.$

Similarly, $7\sqrt{\frac{1}{2}}-\frac{3}{2}\sqrt{2}=\frac{7}{2}\sqrt{2}-\frac{3}{2}\sqrt{2}=2\sqrt{2}.$

The last example illustrates the necessity of acquiring the ability to simplify radicals before attempting to carry out the fundamental operations of addition and subtraction.

If the radicands are unlike and cannot be simplified further, the radicals are really dissimilar terms, and addition or subtraction can only be indicated. (See page 34.)

Thus $\sqrt{2}$, $\sqrt{3}$, and $\sqrt[3]{5}$ are three dissimilar radicals, and the addition of the three can only be indicated thus: $\sqrt{2}+\sqrt{3}+\sqrt[3]{5}$.

EXERCISES

Simplify [redacted]:

1. $\sqrt{27}+\sqrt{12}.$
2. $\sqrt{45}-\sqrt{20}.$
3. $2\sqrt{200}-3\sqrt{8}.$
4. $\sqrt{28}+\sqrt{63}.$
5. $7\sqrt{18}-\sqrt{98}.$
6. $\sqrt{75}-\sqrt{27}+2\sqrt{48}.$
7. $\sqrt{2}+\sqrt{\frac{1}{2}}.$
8. $\sqrt{3}-\sqrt{\frac{1}{3}}.$
9. $\frac{3}{2}+\sqrt{\frac{3}{4}}.$
10. $\frac{2}{3}+\sqrt{\frac{4}{3}}.$
11. $\frac{R}{2}+\sqrt{\frac{9R^2}{2}}.$
12. $\frac{3R}{2}-\sqrt{\frac{9R^2}{2}}.$
13. $R-\sqrt{\frac{3R^2}{4}}.$
14. $5\sqrt{\frac{1}{2}}-\frac{3}{2}\sqrt{2}.$
15. $8\sqrt{\frac{9}{2}}-\sqrt{72}.$
16. $\sqrt{\frac{1}{3}}+2\sqrt{\frac{4}{3}}-3\sqrt{\frac{2}{3}}.$
17. $\sqrt{\frac{2}{5}}-\sqrt{\frac{9}{10}}+2\sqrt{10}.$
18. $\sqrt{\frac{3}{10}}-\sqrt{120}-2\sqrt{\frac{6}{5}}.$
19. $\sqrt{\frac{5}{18}}+2\sqrt{\frac{32}{5}}-\sqrt{\frac{10}{9}}.$
20. $\sqrt[3]{56}+2\sqrt[3]{189}.$
21. $2\sqrt[3]{320}-\sqrt{50}.$
22. $\sqrt[4]{25}-\sqrt{20}.$

HINT. $\sqrt[4]{25}=\sqrt[4]{5^2}=5^{\frac{2}{4}}=5^{\frac{1}{2}}=\sqrt{5}$, etc.

23. $\sqrt[4]{9}+\sqrt{12}$. 24. $\sqrt[4]{32}+5\sqrt[4]{162}$.

25. $2x\sqrt[3]{54x}-3\sqrt[3]{16x^4}+\sqrt[6]{4x^2}$.

26. $\sqrt[3]{81x^7}+x\sqrt[3]{375x^4}-\sqrt[12]{16x^4}$.

27. $\sqrt{a^3bc}-a\sqrt{abc}+ac\sqrt{\frac{b}{ac}}$.

28. $rs\sqrt[3]{rs}+\sqrt[3]{\frac{1}{r^2s^2}}-2\sqrt[3]{r^4s^4}$.

29. $\sqrt{a^8+4a^{[illegible]}+4a}-\sqrt{a^8}-\frac{2}{a^8}\sqrt{a^7}$.

30. $\sqrt{3x^2-18x+27}-\sqrt{27(x^2+2x+1)}$.

Note. Though methods of classifying irrational expressions are found in the work of Euclid, the Hindus and the Arabs were the first to develop this part of algebra in anything like the form used to-day.

111. Multiplication of radicals. Monomial radicals of the same order are multiplied as follows:

EXAMPLES

1. Multiply $3\sqrt{8}$ by $2\sqrt{5}$.

Solution. $3\sqrt{8}\cdot 2\sqrt{5}=6\sqrt{40}=6\cdot 2\sqrt{10}=12\sqrt{10}$.

2. Multiply $5\sqrt[3]{4ax^2}$ by $\sqrt[3]{2a^2x^2}$.

Solution. $5\sqrt[3]{4ax^2}\cdot\sqrt[3]{2a^2x^2}=5\sqrt[3]{8a^3x^4}=10ax\sqrt[3]{x}$.

The method just illustrated of multiplying monomial radicals of the same order may be stated in the

Rule. *Take the product of the coefficients of the radicals for the coefficient of the radical in the result.*

Multiply together the radicands and write the product under the common radical sign.

Reduce the result to its simplest form.

The preceding rule does not hold for radicals of different orders.

EXERCISES

Perform the indicated operation and simplify results:

1. $\sqrt{5}\cdot\sqrt{6}$.

2. $\sqrt{7}\cdot\sqrt{7}$.

3. $2\sqrt{3}\cdot 3\sqrt{2}$.

4. $3\sqrt{2}\cdot 5\sqrt{2}$.

5. $\sqrt{2}\cdot\sqrt{8}$.

6. $\sqrt{3}\cdot\sqrt{27}$.

7. $5^{\frac{1}{2}}\cdot 20^{\frac{1}{2}}$.

8. $18^{\frac{1}{2}}\cdot 8^{\frac{1}{2}}$.

9. $\sqrt{\frac{12}{25}}\cdot\sqrt{75}$.

10. $\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{27}{8}}$.

11. $\sqrt{11}\cdot\sqrt{\frac{1}{11}}$.

12. $a^{\frac{1}{2}}\cdot(bc)^{\frac{1}{2}}$.

13. $R\sqrt{2}\cdot R\sqrt{3}$.

14. $\left(\sqrt{\frac{R}{2}-\sqrt{2}}\right)^2$.

15. $\frac{R}{2}\sqrt{2}\cdot R\sqrt{3}$.

16. $2\sqrt{x}\cdot\sqrt{4x^3}$.

17. $2\sqrt{rs}\cdot 7\sqrt{r^3s^3t^2}$.

18. $\sqrt{75a}\cdot(45a)^{\frac{1}{2}}$.

19. $\sqrt{2u}\cdot\sqrt{4v}\cdot\sqrt{6uv}$.

20. $5\sqrt{3m}\cdot 5\sqrt{3m}$.

21. $(3\sqrt{3x})^2$.

22. $\sqrt[3]{16}\cdot\sqrt[3]{4}$.

23. $\sqrt[3]{4}\cdot\sqrt[3]{12}$.

24. $(100)^{\frac{1}{3}}\cdot(30)^{\frac{1}{3}}$.

25. $\sqrt[4]{8}\cdot\sqrt[4]{32}$.

26. $5\sqrt[3]{2a}\cdot 3\sqrt[3]{16a}$.

27. $(\sqrt[3]{R-\sqrt{2}})^2$.

28. $\left(\frac{R}{2}\sqrt{2-\sqrt{5}}\right)^2$.

29. $\sqrt{\frac{a}{x}}\cdot\sqrt{\frac{4x}{a}}$.

30. $\sqrt{\frac{5x}{a}}\cdot\sqrt{\frac{a^3}{5x^2}}$.

Find the radical expression having the coefficient 1 equivalent to each of the following:

31. $5\sqrt{2}$.

Solution. $5\sqrt{2}=\sqrt{25}\sqrt{2}=\sqrt{50}$.

32. $7\sqrt{3}$.

33. $10\sqrt{5}$.

34. $8\sqrt{\frac{1}{2}}$.

35. $12\sqrt{\frac{5}{3}}$.

36. $a\sqrt{x}$.

37. $2a\sqrt{c}$.

38. $3\sqrt[3]{2}$.

HINT. $3\sqrt[3]{2}=\sqrt[3]{27}\cdot\sqrt[3]{2}$.

39. $5\sqrt[3]{2}$.

40. $2\sqrt[3]{3}$.

41. $2\sqrt[3]{4}$.

42. $3\sqrt[3]{\frac{1}{3}}$.

43. $a\sqrt[3]{\frac{1}{a}}$.

44. $(a+x)\sqrt[3]{\frac{1}{a+x}}$.

The multiplication of binomial or of polynomial radical expressions of the same order involves no new principle.

EXAMPLE

Multiply $3\sqrt{5}-4\sqrt{3}$ by $2\sqrt{5}+\sqrt{3}$.

Solution.
$$\begin{array}{l} 3\sqrt{5}-4\sqrt{3} \\ \underline{2\sqrt{5}+\ \ \sqrt{3}\quad} \\ 30\quad -8\sqrt{15} \\ \underline{\qquad +3\sqrt{15}-12} \\ 30\quad -5\sqrt{15}-12=18-5\sqrt{15}. \end{array}$$

$$\begin{array}{l} 3\sqrt{4}-9\sqrt{2} \\ -12+5\sqrt{2} \\ 6-9\sqrt{2} \\ \underline{-12+5\sqrt{2}} \\ -9-4 \end{array}$$

EXERCISES

Perform the indicated multiplication and simplify the results:

1. $(\sqrt{2}-3)(3\sqrt{2}+5)$.
2. $(2\sqrt{5}-4)(3\sqrt{5}+3)$.

$(4\sqrt{7}-3)^2$.

$(2\sqrt{7}-3\sqrt{2})^2$.

$(3\sqrt{2}+\sqrt{3})(2\sqrt{3}-\sqrt{2})$.

$(2\sqrt{5}-3\sqrt{2})(3\sqrt{5}+2\sqrt{2})$.

$(\sqrt{5}-\sqrt{3}+\sqrt{2})(\sqrt{5}-\sqrt{3}+\sqrt{2})$.

$(3\sqrt{2}+2\sqrt{3}+\sqrt{30})(\sqrt{2}+\sqrt{3}-\sqrt{5})$.

$(R\sqrt{2}-2)(R\sqrt{2}-3)$.

10. $(R\sqrt{3}-R\sqrt{2})(R\sqrt{3}+2R\sqrt{2})$.

11. $\left(R-\frac{R}{2}\sqrt{2}\right)\left(R-\frac{R}{2}\sqrt{2}\right)$.

12. $\left(\frac{R}{2}+\frac{R}{2}\sqrt{3}\right)^2$.

13. $\frac{R}{2}(\sqrt{5}-1)^2$.

14. $R^2-\left(\frac{R}{2}\sqrt{5}-\frac{R}{2}\right)^2$.

15. $R^2-\left(\frac{R}{2}-\frac{R}{2}\sqrt{2}\right)^2$.

16. $\sqrt{R^2-\left(\frac{R}{2}-\frac{R}{2}\sqrt{3}\right)^2}$.

17. $\sqrt{R^2-\left(\frac{R\sqrt{5}-R}{4}\right)^2}$.

18. The hypotenuse of a right triangle is R and one leg is $R - \frac{R}{2}\sqrt{3}$. Find the other leg.

19. One leg of a right triangle is $\frac{R}{2}(\sqrt{5} - 1)$ and the hypotenuse is R. Find the other leg.

20. The legs of a right triangle are $R\sqrt{2 + \sqrt{2}}$ and $\frac{R}{2}\sqrt{2 - \sqrt{2}}$. Find the hypotenuse and the area.

21. Show by substituting and simplifying that $2 + 2\sqrt{3}$ is a root of the quadratic equation $x^2 - 4x = 8$.

22. Does $\frac{7}{2} - \frac{1}{2}\sqrt{41}$ satisfy $x^2 - 7x + 2 = 0$?

112. Radical equations. An irrational or radical equation in one unknown is an equation in which the unknown occurs in a radicand. The solution of the simpler types of radical equations depends mainly on a knowledge of multiplication of radicals. The ability to square or to cube each member of an equation and the exercise of a little judgment *in transposing at the proper time* are sufficient to solve any of the equations which arise in elementary work.

The necessity of checking all results obtained cannot be too strongly emphasized. The pupil should remember that an answer is a *root* on the one condition that it *satisfies* the *original equation.*

EXERCISES

Solve for x and check:

1. $\sqrt{2x - 7} = 3$.

HINT. Squaring each member, $2x - 7 = 9$, etc.

2. $\sqrt{3x - 5} = 5$.

3. $\sqrt{x - 6} = \sqrt{5}$.

4. $\sqrt{2x + 3} = \sqrt{7}$.

5. $2\sqrt{x - 1} = \sqrt{6}$.

6. $\sqrt{x - 3} - 6 = 0$.

HINT. Transpose before squaring.

7. $2\sqrt{3x + 1} - 8 = 0$.

8. $2\sqrt{5x - 4} + 3 = 11$.

9. $\sqrt[3]{x - 1} = 2$.

10. $\sqrt[3]{2x + 1} = 3$.

-72-

11. $\sqrt[3]{4x-6}=4$.

12. $\sqrt[3]{2x-1}+5=0$.

13. $3\sqrt[3]{2x+7}=6$.

14. $2\sqrt[3]{3x-9}+4=0$.

15. $3\sqrt{x-2}=\sqrt{2x+3}$.

16. $2\sqrt{x+3}=\sqrt{x+18}$.

17. $3\sqrt{2x-6}=2\sqrt{x+4}$.

18. $\sqrt{2x-1}-\sqrt{2-x}=0$.

HINT. Transpose, then square.

19. $\sqrt{4x-3}-\sqrt{3x+5}=0$.

20. $\sqrt{6x-5}-\sqrt{5x+7}=0$.

21. $2\sqrt{3x-3}-3\sqrt{2+x}=0$.

22. $3-\sqrt{x+3}=\sqrt{x}$.

HINT. Squaring each member, $9-6\sqrt{x+3}+x+3=x$.
Transposing, $-6\sqrt{x+3}=-12$.
Dividing by -6, $\sqrt{x+3}=2$.
Squaring again, $x+3=4$, etc.

23. $5-\sqrt{2x+5}=\sqrt{2x}$.

24. $7-\sqrt{3x+7}-\sqrt{3x}=0$.

HINT. $7-\sqrt{3x}=\sqrt{3x+7}$, etc.

25. $\sqrt{3x-5}+\sqrt{3x+7}=6$.

26. $\sqrt{x}+\sqrt{2}-\sqrt{x+2}=0$.

27. $\sqrt{x+1}-\sqrt{2x-3}=\sqrt{3x-2}$.

HINT. Squaring, $x+1-2\sqrt{(x+1)(2x-3)}+2x-3=3x-2$.
Transposing, $2\sqrt{(x+1)(2x-3)}=0$, etc.

28. $\sqrt{x-3}+\sqrt{2x+4}=\sqrt{3x+1}$.

29. $\sqrt{x+3}+\sqrt{x-4}=\sqrt{4x-3}$.

30. $\sqrt{x+4}+\sqrt{x-3}=\sqrt{4x-3}$.

31. $\dfrac{\sqrt{x}-3}{\sqrt{x}+1}=\dfrac{\sqrt{x}-4}{\sqrt{x}-2}$.

32. $\dfrac{\sqrt{x}-3}{\sqrt{x}+2}=\dfrac{\sqrt{x}-5}{\sqrt{x}-1}$.

113. Division of radicals. It is frequently necessary to find the approximate value of an expression which involves division by a radical expression. Thus $2 \div \sqrt{3}$, $\sqrt{3} \div \sqrt{5}$, $\frac{4-\sqrt{3}}{2-\sqrt{3}}$, and $\frac{3\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ are types which often occur.

To find the approximate value of $2 \div \sqrt{3}$ we may extract the square root of 3 to several decimal places and then divide 2 by the approximate root obtained. Both of these processes are long and one of them is unnecessary. For, writing $2 \div \sqrt{3}$ as a fraction and multiplying both terms by $\sqrt{3}$ gives $\frac{2\sqrt{3}}{3}$. To find the approximate value of this last fraction requires but one long operation.

Similarly, the process of finding the approximate value of $\sqrt{7} \div (\sqrt{7}-\sqrt{2})$ involves three rather lengthy operations — the extracting of two square roots, and one long division. The labor of two of these operations can be avoided in a manner similar to that shown above.

114. Rationalizing factors. One radical expression is the **rationalizing factor** for another if the *product* of the two is *rational.*

A rationalizing factor of $\sqrt{3}$ is $\sqrt{3}$, for $\sqrt{3} \cdot \sqrt{3} = 3$.

For $\sqrt[3]{2}$ a rationalizing factor is $\sqrt[3]{4}$, since $\sqrt[3]{2} \cdot \sqrt[3]{4} = \sqrt[3]{8} = 2$.

Similarly, $\sqrt{7}-\sqrt{2}$ is a rationalizing factor of $\sqrt{7}+\sqrt{2}$, as their product $(\sqrt{7}-\sqrt{2})(\sqrt{7}+\sqrt{2}) = 7-2 = 5$.

In like manner $(3\sqrt{5}-2\sqrt{3})(3\sqrt{5}+2\sqrt{3}) = 45-12 = 33$. Therefore $3\sqrt{5}-2\sqrt{3}$ is a rationalizing factor of $3\sqrt{5}+2\sqrt{3}$.

The binomial radicals of the last two illustrations are of the general types $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$. Such binomials are called **conjugate** radicals and *either* is a rationalizing factor of the other. If a and b are rational, the product $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$, or $a-b$, is also *rational.*

ORAL EXERCISES

Determine a rationalizing factor for each of the following expressions and find the product of the given expression and this factor:

1. $\sqrt{5}$.
2. $3\sqrt{6}$.
3. $2\sqrt{7}$.
4. $\sqrt{8}$.
5. $\sqrt{32}$.
6. $\sqrt{27}$.
7. $\sqrt[3]{2}$.
8. $\sqrt[3]{3}$.
9. $2\sqrt[3]{5}$.
10. $\sqrt[3]{16}$.
11. $\sqrt[3]{25}$.
12. $\sqrt{2}+3$.
13. $\sqrt{3}-\sqrt{2}$.
14. $3+\sqrt{7}$.
15. $3\sqrt{2}-5$.
16. $4\sqrt{3}-\sqrt{2}$.
17. $2\sqrt{5}+7\sqrt{6}$.
18. $\sqrt{x}-\sqrt{a}$.
19. $\sqrt{3a}+\sqrt{x}$.
20. $\sqrt[3]{a-2x}$.

Direct division of similar radicals, coefficient by coefficient and radicand by radicand, is often possible.

Thus $\sqrt{6}\div\sqrt{2}=\sqrt{3}$.

And $12\sqrt{10}\div 2\sqrt{5}=6\sqrt{2}$.

But $\sqrt{7}\div\sqrt{3}=\dfrac{\sqrt{7}}{\sqrt{3}}=\dfrac{\sqrt{7}\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}=\dfrac{\sqrt{21}}{3}$.

If direct division cannot be exactly performed, we use the

Rule. *Write the dividend over the divisor in the form of a fraction. Then multiply the numerator and denominator of the fraction by the rationalizing factor of the denominator and simplify the resulting fraction.*

EXERCISES

Perform the indicated division:

1. $\sqrt{10}\div\sqrt{2}$.
2. $\sqrt{18}\div\sqrt{3}$.
3. $\sqrt{5}\div\sqrt{2}$.

Solution. $\sqrt{5}\div\sqrt{2}$.

$$\frac{\sqrt{5}}{\sqrt{2}}=\frac{\sqrt{5}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}=\frac{\sqrt{10}}{2}.$$

4. $\sqrt{7}\div\sqrt{3}$.
5. $2\sqrt{6}\div 3\sqrt{2}$.
6. $\sqrt{10}\div\sqrt{3}$.
7. $2\sqrt{7}\div\sqrt{2}$.
8. $\sqrt{6}\div\sqrt{18}$.

9. $6 \div 2\sqrt{2}$.

10. $8\sqrt{15} \div 4\sqrt{5}$.

11. $3\sqrt{2} \div 15\sqrt{8}$.

12. $\sqrt{6} - \sqrt{18} \div \sqrt{2}$.

13. $(\sqrt{12} - \sqrt{24}) \div 3\sqrt{3}$.

14. $(\sqrt{6} - \sqrt{9} + 18) \div 2\sqrt{2}$.

15. $8 \div (3 + \sqrt{7})$.

Solution.

$$8 \div (3 + \sqrt{7}) = \frac{8}{3 + \sqrt{7}} = \frac{8(3 - \sqrt{7})}{(3 + \sqrt{7})(3 - \sqrt{7})}$$
$$= \frac{24 - 8\sqrt{7}}{9 - 7} = 12 - 4\sqrt{7}.$$

16. $9 \div (3 - \sqrt{2})$.

17. $\sqrt{5} \div (\sqrt{5} - \sqrt{2})$.

Hint.
$$\frac{\sqrt{5}}{\sqrt{5} - \sqrt{2}} = \frac{\sqrt{5}(\sqrt{5} + \sqrt{2})}{(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})}, \text{ etc.}$$

18. $\dfrac{2\sqrt{3}}{\sqrt{3} + \sqrt{2}}$.

19. $\dfrac{4\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}$.

20. $\dfrac{\sqrt{3} + \sqrt{7}}{\sqrt{7} - 2}$.

Find to three decimals the value of:

21. $5 + 3\sqrt{2}$.

22. $7 - 3\sqrt{5}$.

23. $\dfrac{7 + \sqrt{6}}{3}$.

24. $\dfrac{\sqrt{2}}{\sqrt{3}}$.

25. $\dfrac{\sqrt{5} + 1}{\sqrt{5}}$.

26. $\dfrac{2}{\sqrt{3} - \sqrt{2}}$.

27. $3\sqrt{8} - \sqrt{7}$

28. $\sqrt{3} - \sqrt{2}$.

Simplify:

29. $\sqrt{a} \div \sqrt{x}$.

30. $\sqrt{3a^2} \div \sqrt{ax^2}$.

31. $2\sqrt{ax} \div 3\sqrt{bx}$.

Biographical Note. *François Vieta.* The reason that algebra is a universal language which does not depend entirely on the nationality of the writer lies in the fact that the symbols used to indicate the various operations and relations are widely understood and adopted. This has not always been the case, and for a long time during the early history of the subject there was no accepted notation in algebra, but each man used any symbol that suited him. One of the men who did most to establish a fixed notation was François Vieta (1540–1603), a French lawyer who studied and wrote on mathematics as a pastime. He was in public life during his whole career and was well known for his ability to decipher the hidden meaning of dispatches captured from the enemy.

Solve $\sqrt{x-3} = x-5$

$\sqrt{10-x^2} = 4-x$

$2+\sqrt{2x+7} = 2\sqrt{\ldots}$ [illegible]

FRANÇOIS VIETA

$5+\sqrt{5}$ [illegible] $1+\sqrt{5}\ \sqrt{5}$

It was he who established the use of the signs + and – for addition and subtraction, which, to be sure, had been used before his time, but were not generally accepted. He also denoted the known numbers in an equation by the consonants, *B*, *C*, *D*, etc., and the unknowns by the vowels, *A*, *E*, *I*, etc. He recognized the existence of negative roots of equations, but rejected them as absurd.

To denote the second and third powers of the unknown, he used the letters *Q* (*quadratus*) and *C* (*cubus*) respectively. Instead of using the sign = he wrote aeq. (*aequalis* or *aequatur*). Thus Vieta would have written the equation $x^3 - 8x^2 + 16x = 40$ in the form

$$1C - 8Q + 16N \text{ aeq. } 40.$$

Before the time of Vieta this equation would have been written in a much more primitive notation. For instance, with writers only a little earlier it would appear as

$$\text{Cubus } \bar{m} \text{ 8Census } \bar{p} \text{ 16 rebus aequatur 40.}$$

It is easily seen that operations on equations in this form would be very hard to perform.

Vieta is further distinguished as being the first man to obtain an exact numerical expression for the number π, which occurs in geometry. His form of expression calls for an infinite number of operations which, of course, could never be performed, but the further one proceeded, the closer would be the approximation obtained. In a certain sense the familiar sign $\sqrt{}$ implies an infinite number of operations, for one can never go through the process of extracting the square root of 2, for instance, and come out even. Vieta's method of denoting π was, however, more involved than this and made use of complicated irrational fractions.

Divide middle term
by 2 then
of x
square what you get.

CHAPTER XXI

QUADRATIC EQUATIONS

115. Solution by completing the square. The quadratic equation was defined on page 137. Before taking up the work that follows, the student should review the method of forming trinomial squares given on page 121.

ORAL EXERCISES

What terms should be added in order to make the following expressions perfect squares?

1. $x^2 + 2x +$ 1
2. $x^2 + 4x +$ 4
3. $x^2 - 10x +$ 25
4. $x^2 - 6x +$ 9
5. $x^2 + 18x +$ 81
6. $x^2 - 18x +$ 81
7. $x^2 + 3x +$ 9/4
8. $x^2 + x +$ 1/4
9. $x^2 - 5x + ?$ 25/4
10. $x^2 + 7x + ?$
11. $t^2 - \frac{2}{3}t + ?$ 1/9
12. $t^2 + \frac{3}{2}t + ?$ 9/16

EXAMPLES

1. Solve $x^2 + 4x - 21 = 0$. (1)

Solution. Transposing, $x^2 + 4x = 21$. (2)

Adding 4 to each member of (2),

$$x^2 + 4x + 4 = 25. \quad (3)$$

Then $(x + 2)^2 = 5^2$. (4)

Extracting the square root of each member of (4),

$$x + 2 = \pm 5. \quad (5)$$

Whence $x = -2 + 5 = 3$,

and $x = -2 - 5 = -7$.

Check. Substituting 3 for x in (1), $9 + 12 - 21 = 0$, or $0 = 0$.
Substituting -7 for x in (1), $49 - 28 - 21 = 0$, or $0 = 0$.

2. Solve $3x^2 - 10x - 8 = 0$. (1)

Solution. Transposing,
$$3x^2 - 10x = 8. \tag{2}$$

Dividing (2) by the coefficient of x,
$$x^2 - \frac{10x}{3} = \frac{8}{3}. \tag{3}$$

Adding $(-\frac{5}{3})^2$ to each member of (3),
$$x^2 - \frac{10x}{3} + \left(-\frac{5}{3}\right)^2 = \frac{8}{3} + \frac{25}{9} = \frac{49}{9}. \tag{4}$$

Then
$$(x - \tfrac{5}{3})^2 = (\tfrac{7}{3})^2. \tag{5}$$

Extracting the square root of each member of (5),
$$x - \tfrac{5}{3} = \pm \tfrac{7}{3}.$$

Whence
$$x = \tfrac{5}{3} \pm \tfrac{7}{3} = 4 \text{ and } -\tfrac{2}{3}.$$

Check. Substituting 4 for x in (1),
$$3 \cdot 4^2 - 10 \cdot 4 - 8 = 0.$$
$$48 - 40 - 8 = 0, \text{ or } 0 = 0.$$

Substituting $-\frac{2}{3}$ for x in (1),
$$3(-\tfrac{2}{3})^2 - 10(-\tfrac{2}{3}) - 8 = 0.$$
$$\tfrac{4}{3} + \tfrac{20}{3} - 8 = 0.$$
$$\tfrac{24}{3} - 8 = 0, \text{ or } 0 = 0.$$

The method of solving a quadratic equation in x illustrated in the preceding examples may be stated in the

Rule. *Transpose so that the terms containing* x *are in the first member and those which do not contain* x *are in the second.*

Divide both members of the equation by the coefficient of x^2 *unless that coefficient is* $+1$.

Then add to both members the square of one half the coefficient of x *(in the equation just obtained), thus making the first member a perfect trinomial square.*

Rewrite the equation, expressing the first member as the square of a binomial and the second member in its simplest form.

Extract the square root of both members of the equation and write the sign ± before the square root of the second member, thus obtaining two linear equations.

Solve the equation in which the second member is taken with the sign + and then solve the equation in which the second member is taken with the sign –. The results are the roots of the quadratic.

Check. *Substitute each result separately in place of x in the original equation. If the resulting equations are not obvious identities, simplify each until it becomes one.*

If, after the equation has been simplified in accordance with the first two directions of the foregoing rule, the term in the first power of x is lacking, that is, the equation is in the form $x^2 = b$, it is often called a *pure quadratic.* Such an equation may be solved immediately by the extraction of the square root of each member.

Hoffman

Thus, if $5x^2 - 3 = 12$, then $5x^2 = 15$, $x^2 = 3$, and $x = \pm\sqrt{3}$.

EXERCISES

Solve by completing the square, where necessary, and check as directed by the teacher.

1. $x^2 - 4x - 32 = 0$.
2. $x^2 - 2x - 15 = 0$.
3. $x^2 = 18$.
4. $x^2 - 25 = 0$.
5. $x^2 - 7x - 18 = 0$.
6. $x(x + 4) - 3(x + 4) = 0$.
7. $y + 6 = y^2$.
8. $4x^2 - 4x - 3 = 0$.
9. $z^2 - 5\frac{1}{2}z + 2\frac{1}{2} = 0$.
10. $\frac{5}{4} - x^2 = 2x$.
11. $2x^2 - 9x + 4 = 0$.
12. $2t^2 - 3t = 9$.
13. $3x^2 - 5 = 70$.
14. $x^2 + 8 = 4x^2 - 40$.
15. $5x = 6x^2 - 14$.
16. $x^2 - \frac{7x}{12} - 1 = 0$.
17. $6t^2 - 13t + 6 = 0$.
18. $42 + 2x^2 = -19x$.

19. Why is not equation (5), Example 1, page 270, written with the sign ± before each member?

20. $2x^2 - 5x + 1 = 0.$ (1)

Solution. Transposing, $2x^2 - 5x = -1.$ (2)

Dividing each member of (2) by 2,

$$x^2 - \tfrac{5}{2}x = -\tfrac{1}{2}. \qquad (3)$$

Adding $(-\tfrac{5}{4})^2$ to each member of (3),

$$x^2 - \tfrac{5}{2}x + (-\tfrac{5}{4})^2 = -\tfrac{1}{2} + \tfrac{25}{16} = \tfrac{17}{16}. \qquad (4)$$

Then $(x - \tfrac{5}{4})^2 = \tfrac{17}{16}.$ (5)

Extracting the square root of each member of (5),

$$x - \tfrac{5}{4} = \pm\sqrt{\tfrac{17}{16}} = \pm \tfrac{1}{4}\sqrt{17}. \qquad (6)$$

Whence $x = \tfrac{5}{4} \pm \tfrac{1}{4}\sqrt{17}.$ (7)

Now $\tfrac{1}{4}\sqrt{17} = \tfrac{1}{4}(4.123+) = 1.031-.$ (8)

Then $\tfrac{5}{4} + \tfrac{1}{4}\sqrt{17} = 1.25 + (1.031-) = 2.281-.$ (9)

Also $\tfrac{5}{4} - \tfrac{1}{4}\sqrt{17} = 1.25 - (1.031-) = .219+.$ (10)

Check. Since (9) and (10) are not the *exact* values of x, they will, if substituted for x in (1), make its first member *nearly* but *not quite* zero. An exact check on the radical forms of the roots can be obtained by substituting from equation (7) in equation (1).

The check may be shortened by substituting both roots at the same time, as follows:

Substituting $\tfrac{5}{4} \pm \tfrac{1}{4}\sqrt{17}$ for x in $2x^2 - 5x + 1 = 0$,

$$2(\tfrac{5}{4} \pm \tfrac{1}{4}\sqrt{17})^2 - 5(\tfrac{5}{4} \pm \tfrac{1}{4}\sqrt{17}) + 1 = 0.$$

$$2(\tfrac{25}{16} \pm \tfrac{5}{8}\sqrt{17} + \tfrac{17}{16}) - 5(\tfrac{5}{4} \pm \tfrac{1}{4}\sqrt{17}) + 1 = 0.$$

$$\tfrac{25}{8} \pm \tfrac{5}{4}\sqrt{17} + \tfrac{17}{8} - \tfrac{25}{4} \mp \tfrac{5}{4}\sqrt{17} + 1 = 0.$$

$x^2 - 4x =$

$x^2 - 4x + 4$

$x - 2$ -4

The radical terms vanish because the two upper signs before them must first be taken together, and then the two lower signs.

Therefore $\tfrac{25}{8} + \tfrac{17}{8} - \tfrac{50}{8} + \tfrac{8}{8} = 0$, or $\tfrac{50}{8} - \tfrac{50}{8} = 0.$

In quadratic equations like the preceding the radical of the roots are often sufficient; at other times values t

$7 - x^2 - x = 3/4$

$x^2 - x + 1/4 =$ ~~11~~

$x - 1/2 = \pm 1$

$x =$

x

or three decimal places are necessary. Unless otherwise directed, obtain only the radical forms of irrational roots.

In the following ten exercises obtain correct to three decimal places the values of any radical answers which may occur:

21. $p^2 + 10p + 17 = 0.$

22. $x^2 + 2x - 4 = 0.$

23. $x^2 - 2x - 3\frac{1}{2} = 0.$

24. $3x^2 - 6x + 1 = 0.$

25. $9x = 5x^2 - 2.$

26. $1 - 4v^2 = 2v.$

27. $x^2 - 6\sqrt{3}x = -9.$

28. $q^2 - 3q\sqrt{2} + 4 = 0.$

29. $x^2 + 5 = 6 - 4x - 3x^2.$

30. $3 + \dfrac{1}{3p} = 3p.$

31. $x^4 - 4x^2 + 3 = 0.$

This is not a quadratic equation, but many equations of this form can be solved by completing the square.

Solution.

$$x^4 - 4x^2 + 3 = 0.$$
$$x^4 - 4x^2 = -3.$$
$$x^4 - 4x^2 + 4 = -3 + 4 = 1.$$
$$x^2 - 2 = \pm 1.$$
$$x^2 = 3 \text{ and } 1.$$
$$x = \pm\sqrt{3} \text{ and } \pm 1.$$

Check as usual.

32. $x^4 - 5x^2 + 4 = 0.$

33. $9x^4 - 13x^2 + 4 = 0.$

34. $9x^4 - 82x^2 + 9 = 0.$

35. $x^4 - 10x^2 + 24 = 0.$

36. $4x^4 - 13x^2 + 3 = 0.$

37. $16x^4 - 81x^2 + 5 = 0.$

NOTE. The student has undoubtedly noticed that quadratic equations have two roots. Mathematicians attempted for many years to prove that cubic equations always have three roots and that equations of the fourth degree have four roots. This was finally done in the seventeenth century by the Italians Cardan and Ferrari.

That any equation in one unknown has a number of roots equal to its degree was first proved by K. F. Gauss. He gave three distinct proofs of this fact, although no one before his time had been able to prove it at all. Since the time of Gauss hundreds of proofs have been given by mathematicians in all parts of the world.

The mathematical researches which have been stimulated and carried to a high degree of completeness by the study of this problem

9, 2
4, -2
-9/5, 7/3

2, -24
1, -1/6
4, -2

-7/2 ± 1/2 √7
4 ± √11

have been very numerous. This situation illustrates the important fact that when a scholar takes up a scientific study with the intention of reaching a definite result, the indirect results of his investigations are frequently even more important than the solution of the main problem which he had set himself.

Few scientists possess the vision to determine surely at the beginning of an investigation all of the directions in which it may lead. The distinction, however, between a great man and a little one largely lies in the instinct which the great man has to treat subjects which prove to be fruitful.

$x^2 - 2x = 4/3$
$x^2 - 2x + 1 = 7/3$

REVIEW EXERCISES

Solve, and check as directed by the teacher:

1. $12x^2 + 7x = 5.$
2. $(5x - 2)(x - 6) = (2x - 12)(2x - 4).$
3. $4x^2 + 15x = -9.$
4. $x + 1 = 20x^2.$
5. $25y^2 - 20y - 12 = 0.$
6. $6x + x^2 + 3 = 2 - 3x^2 + 10x.$
7. $(3x + 6)(x - 3) = (2x + 1)(x - 2).$
8. $\frac{x}{3} + 4 - \frac{15}{x} = 0.$
9. $\frac{1}{3} + \frac{a}{9} - \frac{2}{a} = 0.$
10. $\frac{3x}{2} + \frac{1}{2} = \frac{1}{3x}.$
11. $x = \frac{3}{x + 2}.$
12. $\frac{x^2}{x - 5} + \frac{5}{2} = 0.$
13. $\frac{2}{t - 2} - \frac{3t}{2} = 0.$
14. $\frac{x}{5 - x} + \frac{5x}{12} = \frac{3}{2}.$
15. $\frac{x}{x - 2} - \frac{5}{2} = \frac{2 - x}{x}.$
16. $\frac{2}{s - 2} - \frac{1}{s + 2} = -\frac{15}{8}.$
17. $\frac{3 + x}{4 + x} - \frac{1}{12} = \frac{5 - x}{6 - x}.$
18. $\frac{1 + x}{2 + x} - \frac{1}{6} = \frac{3 - x}{5 - x}.$
19. $\frac{x + 2}{x - 4} + \frac{x - 1}{x + 5} = -1.$
20. $\frac{x - 2}{x + 2} - \frac{x - 1}{x + 1} = 6.$
21. $(3x - 2)(x - 3) = (x + 1)(x + 4).$
22. $x^4 - 10x^2 + 16 = 0.$
23. $x^4 - 8x^2 + 15 = 0.$
24. $6x^4 - 19x^2 + 15 = 0.$
25. $8x^4 - 18x^2 + 9 = 0.$

26

4 + 6

PROBLEMS

(Reject all answers which do not satisfy the conditions of the problems.)

1. The square of a certain number plus twice the number itself is 8. Find the number.

2. If from twice the square of a certain number the number itself be taken, the remainder is 28. Find the number.

3. Find two consecutive odd numbers whose product is 675.

4. Find three consecutive even numbers whose sum is $\frac{1}{2}$ the product of the first two.

5. A rectangular field is 14 rods longer than it is wide. Its area is 20 acres (1 acre = 160 square rods). Find the dimensions of the field.

6. The sum of a certain number and its reciprocal is $2\frac{1}{6}$. Find the number.

7. The area of a triangular field is $4\frac{1}{2}$ acres. The base is 36 rods longer than the altitude. Find the base and the altitude.

8. Two square fields together contain 40 acres. A side of one is 16 rods longer than a side of the other. Find the side of each.

9. The hypotenuse of a right triangle is 41 feet. One leg is 31 feet shorter than the other. Find the legs.

10. One leg of a right triangle is $\frac{3}{4}$ as long as the other. The hypotenuse is 30. Find the legs.

11. The area of a square in square feet and its perimeter in inches are expressed by the same number. Find the side.

12. The dimensions of a certain rectangle and the longest straight line which can be drawn on its surface are represented in inches by three consecutive numbers. Find its dimensions.

13. The edges of two cubical bins differ by one yard. Their volumes differ by 127 cubic yards. Find the edge of each bin.

14. The rates of two trains differ by 6 miles per hour. The faster requires 1 hour less time to run 252 miles. Find the rate of each train.

15. An automobile made a round trip of 180 miles in 11 hours. Returning, the rate was increased 3 miles per hour. Find the rate each way.

16. A page of a certain book is 2 inches longer than it is wide. The printed portion covers half the area of the page and the margin is 1 inch wide. Find the length and width of the page.

17. The price of oranges being raised 10 cents per dozen, one gets 6 fewer oranges for a dollar. Find the original price.

18. Two pumps together can fill a standpipe in 30 minutes. One pump alone requires 32 minutes less time than the other. Find the time each requires alone.

19. How long will it take a stone to reach the ground if dropped from an elevation of 1600 feet? (See Exercise 45, p. 17.)

20. A man drops a stone over a cliff and hears it strike the ground below $6\frac{1}{2}$ seconds later. If sound travels 1152 feet per second, find the height of the cliff.

116. Quadratic equations with literal coefficients. Such equations are solved as in the following example.

EXAMPLE

Solve for x the equation $x^2 - 6ax - 7a^2 = 0$, and check the result.

Solution. Transposing, $x^2 - 6ax = 7a^2$.

Completing the square, $x^2 - 6ax + 9a^2 = 16a^2$.

Then $(x - 3a)^2 = 16a^2$.

Extracting the square root, $x - 3a = \pm 4a$.

Whence, transposing and combining, $x = 7a$, and $-a$.

Check. $(7a)^2 - 6a \cdot 7a - 7a^2 = 49a^2 - 42a^2 - 7a^2 = 0$, or $0 = 0$.

$(-a)^2 - 6a(-a) - 7a^2 = a^2 + 6a^2 - 7a^2 = 0$, or $0 = 0$.

37 C

EXERCISES

Solve for x and check:

41

1. $x^2 + 2bx = 3b^2$.
2. $x^2 + 4cx = 5c^2$.
3. $x^2 - 8cx - 9c^2 = 0$.
4. $x^2 + ax - 2a^2 = 0$.
5. $x^2 - 2ax = 1 - a^2$.
6. $x^2 - a^2 = 0$.
7. $a^2x^2 = b^2$.
8. $x^2 + bx = 2b^2$.
9. $x^2 - 12a = 3ax - 4x$.
10. $2x^2 - 3hx - 2h^2 = 0$.
11. $10x^2 = 3a^2 - ax$.
12. $2x^2 - 5hx + 2h^2 = 0$.
13. $ax^2 - 5x - 2ax + 10 = 0$.
14. $cx^2 + 2x = 3cx + 6$.
15. $ax^2 + a = a^2x + x$.
16. $cx^2 + ax = 2cx + 2a$.

History of the quadratic equation. Though the development of the method of solving quadratic equations is closely connected with the general growth of algebra, yet it is possible to indicate rather briefly the most important steps in the process.

The first writer on formal algebra was Diophantos, who lived at Alexandria, in Egypt, about A.D. 275. Most of his work that is preserved is devoted to the solution of problems that lead to equations. So far as we know he was the first to indicate the unknown number by a single letter, in this respect being far in advance of many mathematicians who lived much later. It is a little remarkable, in fact, that so able and original a man as Diophantos should have exerted so little influence on his successors. He solved his quadratic equations by a method not unlike that of completing the square, but his imperfect knowledge of the nature of numbers made it impossible for him to understand the entire significance of the process. Though he made every effort not to consider equations whose roots were not positive integers, sometimes they would creep in, and under such circumstances, when his method led him to a negative or irrational root, he rejected the whole equation as absurd or impossible. Even when both of the roots were positive he took only the one afforded by the positive sign in the formula for solving a quadratic.

The difficulties of Diophantos are typical of those encountered by mathematicians for the next fifteen hundred years. The difficulty lay not in finding a formal method of solving the equation, but in understanding the result after it was obtained. The meanings of negative and of imaginary numbers have been two of the most difficult of all mathematical ideas for men to grasp.

Five or six hundred years later the Hindus devised a general solution of the quadratic, but their chief advance over Diophantos lay in the fact that they did not regard an equation whose roots were negative as necessarily absurd, but merely rejected the negative result with the remark, "It is inadequate; people do not approve of negative roots." The Hindus, however, did realize that a quadratic equation sometimes has two roots, a fact that Diophantos never comprehended.

No material gain in the understanding of the solutions of the quadratic can be found until the seventeenth century. The keenest mathematicians of the sixteenth century, like Cardan and Vieta, rejected negative roots, though by this time irrational roots were admitted. In fact, in 1544 Stifel, a German, published an algebra in which irrational numbers are included among the numbers proper. But he affirms that except in the case where a quadratic equation has two positive roots no equation has more than one root. It was not until the work of Descartes and Gauss became widely known that the nature of the roots of all kinds of quadratic equations was completely understood.

117. Systems involving a linear and a quadratic equation. A **quadratic equation** in two unknowns contains one or more terms of the second degree, but no term of higher degree in those unknowns. Every system of equations in two unknowns in which one equation is **linear** and the other **quadratic** *can be solved by the method of* **substitution.**

EXAMPLE

Solve the system
$$\begin{cases} x^2 + y^2 = 5, & (1) \\ x - y = 1. & (2) \end{cases}$$

Solution. Solving (2) for x, $$x = 1 + y. \qquad (3)$$

Substituting $1 + y$ for x in (1), $$(1 + y)^2 + y^2 = 5. \qquad (4)$$

From (4), $$y^2 + y - 2 = 0. \qquad (5)$$

Solving (5), $$y = 1 \text{ or } -2.$$

Substituting 1 for y in (3), $x = 1 + 1 = 2.$

Substituting -2 for y in (3), $x = 1 - 2 = -1.$

Therefore $\begin{cases} x = 2 \\ y = 1 \end{cases}$ and $\begin{cases} x = -1 \\ y = -2 \end{cases}$ are the two sets of roots.

Check. Substituting 2 for x and 1 for y in $\begin{cases} (1), & 4 + 1 = 5, \\ (2), & 2 - 1 = 1. \end{cases}$

Substituting -1 for x and -2 for y in $\begin{cases} (1), & 1 + 4 = 5, \\ (2), & -1 + 2 = 1. \end{cases}$

The similarity between this method and that for the solution of linear systems by substitution should be carefully noted.

The method of the preceding solution is stated in the

Rule. *Solve the linear equation for one unknown in terms of the other.*

Substitute this value in the quadratic equation and solve the resulting equation.

Substitute each of the roots of the quadratic equation thus found in the linear equation, and solve, thus obtaining two sets of roots of the simultaneous system.

Check, as usual, in both equations.

EXERCISES

Solve the following systems, pair results, and check each set of roots:

1. $x^2 + y = 8,\quad 2x + y = 5.$
2. $x + y = 5,\quad x^2 + y^2 = 13.$
3. $x^2 + y^2 = 25,\quad 3x - 4y = 0.$
4. $x^2 + y^2 = 25,\quad x - 3y = 5.$
5. $xy = 12,\quad 2x + y - 10 = 0.$ 3, 4 2, 6
6. $x^2 + 3xy = 25,\quad 2x + y = 10.$ 5, 0
7. $4p + 5q = 6,\quad pq + 2 = 0.$
8. $4x - 3y = 30,\quad xy + 12 = 0.$
9. $3A + 2B = 5,\quad AB + 3 = 6A.$
10. $x^2 + xy = 5,\quad x - y = 9.$
11. $x^2 + y^2 + y = 46,\quad 2x + y = 10.$
12. $x^2 + 3xy + y^2 = 22,\quad 2x = y.$
13. $x^2 + y^2 - 4x - 2y = 20,\quad 3x - y = 10.$

27

PROBLEMS

(Reject all results which do not satisfy the conditions of the problem.)

1. Find two numbers whose difference is 3 and the difference of whose squares is 45.

2. The sum of two numbers is 18 and the sum of their squares is 170. Find the numbers.

3. A rectangular field is 39 rods longer than it is wide and its area is 10 acres. Find the length and the width.

4. The difference of the areas of two squares is 208 square feet and the difference of their perimeters is 32 feet. Find a side of each square.

5. The perimeter of a rectangle is 92 feet and its area is 504 square feet. Find the length and the width.

6. The base of a triangle is 5 inches longer than its altitude. Its area is $1\frac{1}{4}$ square feet. Find the base and the altitude of the triangle.

7. The perimeter of a rectangle is $3c$ and its area is $\frac{c^2}{2}$. Find its dimensions.

8. Do positive integers differing by 3 exist such that the sum of their squares is 115? If so, find them.

9. If a two-digit number be multiplied by the sum of its digits, the product is 576. If three times the sum of its digits be added to the number, the result is expressed by the digits in reverse order. Find the number.

10. The annual income from a certain investment is $48. If the principal were $200 more and the rate of interest 1% less, the annual income would be $2 more. Find the principal and the rate.

11. A wheelman leaves A and travels north. At the same time a second wheelman, who travels 50% faster than the first, leaves a point 3 miles east of A and travels east. An hour after starting, the distance between them is 17 miles. Find the rate of each.

CHAPTER XXII

REVIEW OF FIRST YEAR ALGEBRA

On the student's return to the study of algebra after a considerable interval of time, some of the principles, definitions, examples, and processes studied in the first year's work may be imperfectly recalled or perhaps wholly forgotten. In the review material which follows, many questions are included as oral exercises. If the student cannot answer these or other questions pertaining to the review, it then becomes an essential part of his work to look up in the preceding pages the points involved and to master them.

FUNDAMENTAL OPERATIONS

118. Order of fundamental operations. Review section 10 and the examples on page 14.

EXERCISES

1. In what order is an indicated series of additions, subtractions, multiplications, and divisions carried out?

Simplify:

2. $6 - 2 + 8 - 7$.

3. $8 + 12 \div 3 - 5$.

4. $8 \cdot 6 \div 4$.

5. $24 \div 2 \div 3$.

6. $36 \div 4 \cdot 3 - 8 + 2$.

7. $(19 - 3 \cdot 5)(8 - 5) \div (14 \div 7)$.

8. $50 - 4(20 - 4 \cdot 2) \div 3 + 2 \cdot 7$.

9. $12 - 8 \div 2 + 10 \cdot 3 - 6 + 8 \cdot 2$.

10. $(28 \div 14 \cdot 24 \div 8 - 3 + 10)(30 \div 3 \div 5 - 2)$.

11. $(16 \div 32 \times 48 \div 8 - 8 + 3) + (42 \div 6 \cdot 7 - 42 - 6) \cdot 6$.

In connection with the rule of section 10 it should be noted that

If the multiplication of two numbers or number symbols is indicated by juxtaposition without any symbol of multiplication, it is customary to think of the multiplication as already having been performed. Thus, $9a \div 7a^2 = \frac{9a}{7a^2}$.

119. Addition and subtraction. Review the definitions, examples, and rules of Chapters III and V.

EXERCISES

1. What is the result of addition called?

2. How are positive numbers added? negative numbers? numbers having unlike signs?

3. What is the name of each of the three numbers in subtraction? Illustrate.

4. What are similar terms? Illustrate.

5. What is a monomial? a binomial? a polynomial?

6. How can the correctness of the result of subtraction be checked?

Add:

7. $12, -8, +4, -3$, and 6.

8. $4a, 3a, -7a, 6a$, and $-2a$.

9. $5a - 3c + 6$, $2a - 6c + 11$, and $4a - c - 9$.

10. $3s - 4t + 6$, $-7t + 6s - 8$, and $t + s + 10$.

11. $a^2 - 3a + 1$, $-2a^2 - 7a + 6$, and $3a^2 - 4 + 5a$.

Subtract the second expression from the first:

12. $6a, 4a$.

13. $8a^3, 15a^3$.

14. $4x + 3, 8x + 6$.

15. $7x^2 - 10, 5x^2 + 20$.

16. $5x - 6, 2x + 8$.

17. $x^2 - 5x + 6, 2x^2 + 8x - 10$.

18. $a^2 - 4ac - 3c^2, 4a^2 + 10ac - c^2$.

19. $3a - 2b - 6c, 4a + 6b - 7c - 2$.

20. $3a^2 - 2c^2 - 6ac$, $5a^2 + 4c^2 - 3ac$.

21. $x - 3y^2 + z - 4ac + 7ax$, $4x - y^2 + 8 - 5ax + 9ac$.

22. $a^3 - c + 3x - a^2m - 8ac$, $4a^3 + m - 8x - 10ac + 4a^2m$.

Find the expression which added to the first will give the second:

23. $3x^2 - 5x + 2$, $6x^2 - 11x + 8$.

24. $4x^2 - 3cx + c^2$, $10x^2 + 8cx - 9c^2$.

Find the expression which subtracted from the second will give the first:

25. $4a^2 - 2ab + b^2$, $7a^2 - 10ab + 6b^2$.

26. $c^2 - 6cx + 8x^2$, $9x^2 - 10cx + 4 + c^2$.

27. From the sum of $t^2 - 4t - 9$ and $3t^2 - 8t + 1$ take the sum of $3t - 6 + 4t^2$ and $5 - 8t^2 + 4t$.

28. From the sum of $ax - ac - 4c^2$ and $4c^2 - 3ac$ take the sum of $4c^2 - 8ax + a^2$ and $4ac + 3ax - 5c^2$.

120. Multiplication. Review the definitions, examples, and processes of Chapter VIII.

EXERCISES

1. Illustrate and define: exponent, coefficient, power, absolute value of a number.

2. What is a literal exponent? Illustrate.

3. What is the name of each of the three numbers in multiplication? Illustrate.

4. What is meant by arrangement of an algebraic expression with respect to a certain letter?

5. Arrange $a^4 + b^4 - 4a^3b - 6a^2b^2 + 4ab^3$ according to the descending powers of b.

6. Arrange $t^3 - 3t^4 - 5 + t - 2t^2$ according to the ascending powers of t; according to the descending powers of t.

7. Is arrangement of multiplier and multiplicand in the same order desirable? Why?

8. In multiplication of monomials how is the coefficient of the product determined? the sign of the product? the exponent of each letter in the product?

9. Using literal exponents, state in symbolic form the law governing the exponents of the like letters in multiplier, multiplicand, and product.

10. How can the correctness of the result of multiplication be checked?

Perform the indicated multiplication:

11. $(6x^2 - 3x + 4)3x$. **14.** $(x^2 - x + 2)(3x - 4)$.

12. $(3x - 5)(4x + 3)$. **15.** $(2x^2 - 5x + 3)(x^2 - 5x + 6)$.

13. $(2s - 3t)(4s + 5t)$. **16.** $(a^4 + 2a^2 - 4)(a^2 - 2a - 3)$.

17. $(t^3 - 2t + 6)(t^4 - 3t^3 - t^2)$.

18. $(2s^2 - 3st + t^2)(s^2 + 5st - 4t^2)$.

19. $(a^2 - ac + c^2)(a^2 + c^2 + ac)$.

20. $(a^2 + 2ab + b^2)(a^2 + b - ab)$.

21. $(4h^2 + 6k^2 + 9hk)(4h^2 + 6k^2 - 9hk)$.

22. $(2a^2 - 3c^6 + 4ac^3)(2a^2 - 3c^6 - 4ac^3)$.

23. $(t^2 + 1 + t)(t^4 + 1 - t^2)(t^2 + 1 - t)$.

24. $(x^2 + 9y^2 + 16 + 3xy - 4x + 12y)(x - 3y + 4)$.

Find the value of:

25. $3x^2 + 2x + 5$ if $x = 5$.

26. $3t^2 - 4t + 8$ if $t = -3$.

27. $9 - 8t + 5t^2 - 3t^3$ if $t = 2$.

28. $2a^3 - 3a^2c + 4ac^2 - 3c^2$ if $a = 3$ and $c = -2$.

29. Does $15(x - a) - 6(x + a) = 3(5a - 3x)$ if $x = 2a$?

30. Does $ax(a + 3) + a(10 - a^2) = x + 3$ if $x = a - 3$?

31. Does $\dfrac{3}{x-1} - \dfrac{2x}{x+1} = \dfrac{3 + 5x - 2x^2}{x^2 - 1}$ if $x = 0$? if $x = -2$?

121. Division. See Chapter X for examples and rules.

EXERCISES

1. In division of monomials how is the coefficient of the quotient determined? the sign of the quotient? the exponent of the letters in the quotient?

2. Using literal exponents, state in symbolic form the law governing the exponents of like letters in multiplication and division.

3. What is the name of each of the three numbers in division? Illustrate.

4. How can the correctness of work in division be checked?

Perform the indicated division:

5. $(2x^2-5x+3)\div(2x-3)$.

6. $(6x^2-13x+6)\div(3-2x)$.

7. $(6a^2+23at-55t^2)\div(3a-5t)$.

8. $(6a^3+6a^2-28-26a)\div(2a+4)$.

9. $(6x^6-5x^4+25x^3-17x^5)\div(5x^2-2x^3)$.

10. $(s^6-7s^3-8)\div(s^2+4+2s)$.

11. $(x^3-5a^2x+2a^3)\div(x^2+2ax-a^2)$.

12. $(2t^4-12t^2-2+11t-7t^3)\div(1-3t-2t^2)$.

13. $(32x^4-60-2x-104x^3+92x^2)\div(5+6x-4x^2)$.

14. $(a^2-2ab+b^2-9x^2)\div(3x+b-a)$.

15. $(4bc+1-4c^2-b^2)\div(2c-b-1)$.

16. $(27a-18ab^2-3a^9+3ab^4)\div(3-b^2+a^4)$.

17. $(3x^5+9x^2+2-5x-8x^3-x^4)\div(x-1)^2$.

18. $(4a^2+9-4ab+b^2-6b+12a)\div(b-3-2a)$.

19. $(x^3+8y^3+125-30xy)\div(x+2y+5)$.

20. $(x^3+y^3+z^3-3xyz)\div(x+y+z)$.

122. Removal and insertion of parentheses. See Chapter VII for examples, principles, and rules.

EXERCISES

Remove parentheses and simplify:

1. $a + (a - b) - (a - 3b)$.
2. $a - (a - c) + (2c - 3a)$.
3. $3b - [(a - b) - (c - a)]$.
4. $2c - 2(a - c) + 3(c - a)$.
5. $x - y + 3(x - y) - 4(2x - y)$.
6. $4x - a + [-(3c - x) - (2a - 3x)]$.
7. $a - [-(a - 3) + (3c - 2a) - 5a] + 6c$.
8. $x - 3 - (a - 2x) - [2(a - x - 5) - 3(6 - 2a)]$.
9. $(2t^2 - 3t) - 2t(3 + t)$.
10. $(a + b)a - (a - b)b + 3ab$.
11. $4x^2 - 3a^2 - (a - 2x)(3x - a)$.
12. $(x - 4)(x - 3) - (x - 3)(x + 2)$.
13. $3a(a - b) - (a + b)(a - b)$.
14. $(x - 3)(x - 4) - (x - 5)(x + 3)$.
15. $6x - [-(a - c) + (3c - 4a)]$.
16. $7c - [(3c - 4) - 6 - (4x - 3a - c)]$.
17. $4x - 2(x - 3) - 3[x - 3(4 - 2x) + 8]$.
18. $3x - 2[1 - 3(2x - 3 - a) - 5\{a - (3x - 2a) - 4\}]$.

Without changing their values, rewrite the following trinomials so that the last two terms are in a parenthesis (*a*) preceded by a plus sign, (*b*) preceded by a minus sign.

19. $x - y - 3$.
20. $2x + y - 8$.
21. $7x - 4y + 10$.
22. $x + y + z$.
23. $x - 3y + 4z$.
24. $3x + 4y - 7z$.
25. $2x + 5y - z$.
26. $3x - 8y + 6z$.
27. $x - y - z$.

123. Important special products. Certain products are of frequent occurrence. These forms should be learned so that one can write or state such results without the labor of actual multiplication. In addition to the four special products of pages 105–111 the three special products which follow should be mastered.

The square of the polynomial $a+b-c$ gives the formula

$$(a+b-c)^2=a^2+b^2+c^2+2ab-2ac-2bc.$$

This may be expressed in words as follows:

The square of any polynomial is equal to the sum of the squares of each of the terms plus twice the algebraic product of each term by every term that follows it in the polynomial.

The cube of the binomial $a+b$ gives the formula

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

This may be expressed in words as follows:

The cube of the sum of two numbers equals the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.

Similarly,

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$$

This can be expressed verbally in a similar manner.

ORAL EXERCISES

State the result of the indicated multiplication:

1. $(x+5)^2$.
2. $(3x+2)^2$.
3. $(x-3)^2$.
4. $(3x-5)^2$.
5. $(x+7a)^2$.
6. $(2x-3y)^2$.
7. $(x^2-x)^2$.
8. $(a^2+3a)^2$
9. $(x-5)(x+5)$.
10. $(x+3)(x+4)$.
11. $(x+1)(x+7)$.
12. $(x-3)(x-5)$.
13. $(x-3)(x+4)$.
14. $(2x-1)(2x+1)$.
15. $(4x-3c)(4x+3c)$.
16. $(ax-c)(ax+c)$.
17. $(3a+2x)(2x-3a)$.

18. $(x+2)(2x-9)$.

19. $(ax-3)(ax+5)$.

20. $(3x+1)(3x+4)$.

21. $(a^2-3a)(a^2+4a)$.

22. $(a+b+c)^2$.

23. $(a+c-x)^2$.

24. $(a-c+x)^2$.

25. $(a-c+1)^2$.

26. $(a+c+6)^2$.

27. $(a-3x+2)^2$.

28. $(a-3b-c)^2$.

29. $(a+b+c+d)^2 = a^2+b^2+c^2+d^2 + 2ab+2ac+2ad + 2bc+2bd + 2cd$.

30. $(a+c+x+d)^2$.

31. $(a+b+c-x)^2$.

32. $(x-y-c-a)^2$.

33. $(x+2a+c+y)^2$.

34. $(3x+a+y-c)^2$.

35. $(2a-c-3x+y)^2$.

36. $(a+c)^3$.

37. $(a+2)^3$.

38. $(a-x)^3$.

39. $(a-2)^3$.

40. $(1-a)^3$.

41. $(a+2x)^3$.

42. $(3a-x)^3$.

43. $(a^2-a)^3$.

44. $(x^2+2x)^3$.

45. $(2a-3x)^3$.

LINEAR EQUATIONS IN ONE UNKNOWN

124. Definitions. For definitions and illustrations of an equation and of an axiom see pages 39–40; of an identity, of an equation of condition, and of the root of an equation see pages 54–55. The foregoing ideas are used in the solution of the following:

EXAMPLE

Solve $\frac{5x}{2} - \frac{1}{2} = x + 7$.

Solution. $\frac{5x}{2} - \frac{1}{2} = x + 7$. (1)

Multiplying each member of (1) by 2, (Ax. III)

$$5x - 1 = 2x + 14. \quad (2)$$

Adding 1 to each member of (2), (Ax. I)

$$5x = 2x + 15. \quad (3)$$

Subtracting $2x$ from each member of (3), (Ax. II)

$$3x = 15. \tag{4}$$

Dividing each member of (4) by 3, (Ax. IV)

$$x = 5. \tag{5}$$

Check. Substituting 5 for x in (1), we have

$$\frac{25}{2} - \frac{1}{2} = 5 + 7, \text{ or } 12 = 12.$$

Since substituting 5 for x satisfies (1), 5 is the root of (1).

125. Transposition in equations. For the definition and illustrations of transposition see pages 56–57.

126. Equivalent equations. Two or more equations in one unknown, even if of very different form, are **equivalent** if all are satisfied by every value of the unknown which satisfies any one of them.

Equations (2), (3), (4), and (5) of section 124 are each equivalent to equation (1) and to each other, for *all are satisfied by the same value of the unknown.*

Of the four axioms or assumptions of section 27 we make constant use. If the "same number" referred to in each axiom does not contain an unknown, the result is always an equation *equivalent* to the *original* one. Further, *if identical expressions involving the unknown* be added to or subtracted from each member of an equation, the resulting equation is equivalent to the first. If, however, both members of an equation be multiplied by identical expressions containing the unknown, the resulting equation *may not* be equivalent to the original one.

Multiplying each member of the equation $x - 2 = 3$ by $x - 1$, we get $x^2 - 3x + 2 = 3x - 3$, or $x^2 - 6x + 5 = 0$. Now this last equation has the roots 1 and 5, whereas the given equation has the root 5 only. Here the root 1 was introduced by multiplying the given equation by $x - 1$. Results obtained from the use of Axiom III with multipliers which contain an unknown should always be carefully checked. When a root is obtained which does not satisfy the original equation, this root should be rejected.

The use of Axiom IV when the divisor contains the unknown may result in the loss of a root which the process of checking will not discover. *If an equation is divided by a factor containing an unknown, this factor should be set equal to zero.* The root of the equation thus obtained is a root of the given equation.

For example, if each member of $x^2 - 4 = 3x + 6$ is divided by $x + 2$, the result is $x - 2 = 3$, whence $x = 5$. Since the equation $x^2 - 4 = 3x + 6$ is satisfied by $x = 5$, then 5 is a root of it.

But $x = -2$ satisfies $x^2 - 4 = 3x + 6$. This root would be lost by dividing by $x + 2$ and solving the resulting equation.

With these and certain other rare exceptions which will be noted later, the application of the axioms will produce an equation equivalent to the given one.

For solving equations in one unknown which do not involve fractions we have the

Rule. *Free the equation of any parentheses it may contain.*

Transpose and solve for the unknown involved.

Reject all values for the unknown which do not satisfy the original equation.

127. Checking. For checking the values obtained from the solution of an equation see page 42.

EXERCISES

Solve, and check the results as directed by the teacher:

1. $5x + 1 = 2x + 7$.

2. $1 + x + 5x + 17 = 0$.

3. $15x = 3(4x - 5)$.

4. $2(x + 2) - (x + 5) = 0$.

5. $3(2x - 7) - 2(5 - 2x) + 1 = 0$.

6. $3(x + 2) - 5(2x - 3) = 0$.

7. $4(4x-1)+3-2(3+x)=0.$

8. $4(x-2)+3(2-x)-3x=6(x+1).$

9. $(x+1)(x-2)=x^2+3.$

10. $(4x-3)(2x-5)-(4x-7)(2x-1)=0.$

11. $(x+2)^2+48=(x-4)^2.$

12. $(x+4)^2-(2-x)^2=84.$

13. $(x+3)(6x+5)-(2x+4)(3x-8)=38.$

14. $(y-4)(6-y)+(y+2)(y-4)=0.$

15. $(x+4)(x+3)=(x+2)(x+1)+42.$

16. Give an illustration of a numerical identity.

17. Describe the process of transposing a term from one member of an equation to the other.

18. What is the essential difference between an identity and an equation of condition?

19. $x-2a=4a-x.$

20. $8s-x=x-4r.$

21. $ax-2ab=4ab-ax.$

22. $m(x-5a)-3m(a-x)=0.$

23. $x(b+a)=ab+a^2.$

24. $x(a-c)=a^2-c^2.$

25. $ax-2cx=a^2-4c^2.$

26. $2ax-4a^2+4a=1+x.$

27. $ax+1-a^3=x.$

28. $(x+a)(x+b)=x^2+2a^2+3ab.$

29. $(a-c)(x-m)=(m-c)(x-a).$

30. $a^2x-a^3+3ax=3-10a+x.$

31. $c(1+x)+m(x+1)-x(m+c+1)=0.$

32. Form an equation in x which has the root 9; the root -15.

128. The solution of problems. Review the steps of a solution as stated and illustrated on pages 45–46.

PROBLEMS

1. One number is five times another, and their sum is 102. What are the numbers?

2. The sum of two numbers is 54. Twice one of them equals ten times the other. What are the numbers?

3. The sum of three consecutive even numbers is 1044. Find the numbers.

4. Of four consecutive numbers the product of the second and fourth exceeds the product of the first and third by 201. Find the numbers.

5. One pupil is four years older than another. Eight years ago the first was twice as old as the second. Find their ages now.

6. Two men are 48 and 18 years of age respectively. How many years hence will the older be twice as old as the younger?

7. One man is three times as old as another. Fifteen years ago the first was six times as old as the second. Find their ages now.

8. A is twice as old as B, and C is three times as old as D. B is 8 years older than D. In ten years the sum of their ages will be 113 years. How old is each?

9. A certain square plot is surrounded by a border 6 feet wide. The area of this border is 816 square feet. What is the side of the square?

10. A certain picture is 4 inches longer than it is wide, and the frame is 2 inches wide. The area of the framed picture is 192 square inches greater than that of the picture alone. What are the dimensions of the picture?

11. Can a rectangle of perimeter 112 inches be drawn which has a length 5 inches greater than twice the width? If so, find its dimensions.

12. A sum of $15.75 consists of dollars, quarters, and dimes. If there are 6 more quarters than dollars, and twice as many dimes as quarters, find the number of coins of each kind.

13. A collection of 109 coins is made up of quarters, dimes, and nickels. There are 7 fewer dimes than quarters, and 3 less than five times as many nickels as dimes. Find the amount of the collection.

14. The sum of the digits of a certain two-digit number is 14. If the order of the digits is reversed, the number is decreased by 36. Find the number.

15. The digits of a certain three-digit number are consecutive odd numbers. If the sum of the digits is 15, find the number.

Facts from Geometry. The area of a circle is the square of the radius multiplied by π ($\pi = \frac{22}{7}$ nearly), or $A = \pi R^2$.

The circumference of a circle equals the diameter times π. The usual formula is $C = 2\pi R$.

16. If the radius of a given circle is increased 14 inches, the area is increased 1232 square inches. Find the first radius.

17. By adding 2 inches to the radius of a circle whose radius is 8 inches, how much is the circumference increased? the area?

18. Substitute R inches for 8 inches in Problem 17 and solve. Interpret your results.

19. Imagine that a circular hoop 1 foot longer than the circumference of the earth is placed about the earth so that it is everywhere equidistant from the equator and lies in its plane. How far from the equator will the hoop be?

20. Compare the result of Problem 19 with the one obtained when a similar process is carried out with a sphere 18 inches in diameter, instead of with the earth.

21. The rope attached to the top of a flagpole is 5 feet longer than the pole. The lower end of the rope just reaches the ground when taken to a point 25 feet from the base of the pole. Find the height of the pole.

FACTORING

129. Definition of factoring. **Factoring** is the process of finding the two or more algebraic expressions whose product is equal to a given expression.

In multiplication we have two factors given and are required to find their product. In division we have the product and one factor given and are required to find the other factor. In factoring, however, the problem is a little more difficult, for we have only the product given, and our experience in multiplication and division is called upon to enable us to determine the factors.

130. Rational expressions. An algebraic expression which can be written without the use of indicated roots of the letters involved is **rational** in these letters.

Thus 2, $5x$, $3y-\sqrt{2}$, and a^2 are rational expressions with respect to the letters involved. In this chapter factors which involve radicals (literal or numerical) will not be sought.

131. Integral expressions. If a rational expression can be written so as not to involve an indicated division in which an unknown letter occurs in a denominator, it is said to be **integral.**

Thus 3, $7a$, $\frac{a}{5}$, and $4x-3$ are integral expressions with respect to the letters involved. In this chapter factors which involve fractions will not be sought.

132. Prime factors. An integral expression is **prime** when it is the product of no two rational integral expressions except itself and 1.

It must be remembered that to factor an integral expression means to resolve it into its prime factors.

The methods of this chapter enable one to factor integral rational expressions in one letter which are not prime, as well as some of the simpler expressions in two letters. No attempt is made even to define what is meant by prime factors of other expressions.

133. Polynomials with a common monomial factor. Study the examples on page 116.

ORAL EXERCISES

Factor:

1. $3a + 6.$
2. $5x + 15.$
3. $a^2 + a.$
4. $9x^2 - 3x.$
5. $cd + c^2d^2.$
6. $h - 21h^2k.$
7. $2c + 4c^2 - 2cd.$
8. $4a - 10a^2 - 2a^3.$
9. $6ac - 3bc + 3c.$
10. $14a^4 - 7a^3 + 7a^2 - 7a.$
11. $3c^2 - 6c^4 + 9c^3 - 15c^4.$
12. $10x^3y^3 - 2xy^3 + 2x^2y^3 - 2x^4y^4.$
13. State the method you have employed in factoring the foregoing exercises.

134. Polynomials which may be factored by grouping terms and taking out a common binomial factor. Study the examples on page 117.

EXERCISES

Separate into polynomial factors:

1. $2(a + b) + x(a + b).$
2. $3(b + 5) + a(b + 5).$
3. $a(c - d) - b(c - d).$
4. $2(x - y) - x(x - y).$
5. $h(m + 3n) - 2k(m + 3n).$
6. $x(r - s) + y(s - r).$

HINT. Write in the form

$$x(r - s) - y(r - s).$$

7. $5r(3m - 2n) - 2s(2n - 3m).$
8. $ac + 2cx + 3ay + 6xy.$
9. $cd - 3cf + 2d - 6f.$
10. $r^2s + 2rs - 3r^2t - 6rt.$
11. $x^3 - 3xy^2 - 3x^2y + 9y^3.$
12. $ax + ay + bx + by + cx + cy.$
13. $mr - 2r + ms - 2s + mt - 2t.$
14. State the method you have employed in factoring the foregoing exercises.

135. Trinomials which are perfect squares. Study the examples and rules on pages 121–122.

ORAL EXERCISES

Separate into binomial factors:

1. $a^2 + 2ax + x^2$.

2. $x^2 - 2xy + y^2$.

3. $4 - 4x + x^2$.

4. $r^2 - 10rs + 25s^2$.

5. $16 - 8ax + a^2x^2$.

6. $(a + b)^2 - 2(a + b) + 1$.

7. $(r - s)^2 - 6(r - s) + 9$.

8. $9 + 6(a + x) + (a + x)^2$.

9. $16 - 8(a - 2x) + (a - 2x)^2$.

10. $(a + b)^2 - 6(a + b)(c - d) + 9(c - d)^2$.

11. $a^{2n} - 12a^n + 36$.

12. $x^{2a} - 14x^ay^b + 49y^{2b}$.

13. State the method employed in factoring the foregoing exercises.

136. A binomial which is the difference of two squares. Study the examples, solutions, and hints on pages 124 and 126.

EXERCISES

Factor:

1. $a^2 - x^2$.

2. $x^2 - 9$.

3. $16 - x^2$.

4. $a^2 - 16b^2$.

5. $16x^2 - 25y^2$.

6. $25a^2 - 36b^2c^2$.

7. $a^4 - 25b^4$.

8. $x^4y^4 - 9$.

9. $a^4 - 16$.

10. $a^4 - b^4$.

11. $x^4 - 1$.

12. $x^8 - 1$.

13. $16 - a^4$.

14. $625 - x^4$.

15. $81 - c^4$.

16. State the method employed in factoring the foregoing exercises.

17. $a^4 - x^8$.

18. $(a - 2)^2 - c^2$.

19. $16(x - y)^2 - z^4$.

20. $9(x - 3y)^2 - 16z^4$.

21. $a^2 - (b + c)^2$.

22. $4r^2s^2 - (r - s)^4$.

23. $9m^2 - 4(n + 3)^4$.

24. $(a - c)^2 - (d + e)^2$.

25. $a(r + 2s)^2 - a(x - y)^2$.

26. $4a(a-x)^2-9a(c-2d)^4$.

27. $m^2+2mn+n^2-(x^2-2xy+y^2)$.

28. $x^2-2xy+y^2-a^2-2ab-b^2$.

29. $a^2-22a+121-b^2+20bc-100c^2$.

30. $4c^2-a^2-2ab-b^2$.

31. $49x^4-49y^2-14y-1$.

32. $a^2-b^2-(a-b)$.

33. $r^2-rs-(r^2-s^2)$.

34. $m+n-m^2+n^2$.

35. $r^2-r-3s-9s^2$.

137. Quadratic trinomials. Study carefully the examples on page 128.

EXERCISES

Separate into binomial factors:

1. x^2+5x+6.

2. $x^2+7x+12$.

3. $d^2+11d+18$.

4. $c^2-2c-35$.

5. r^2-r-90.

6. $9-10x+x^2$.

7. $r^2-4rs+3s^2$.

8. $m^2-7mn+10n^2$.

9. $1+2n-24n^2$.

10. $(a+b)^2-2(a+b)-8$.

11. $(x-y)^2+4(x-y)+3$.

12. $a^{2n}+12a^n+35$.

13. $c^{2a}-7c^a-18$.

14. State the method employed in factoring the foregoing exercises.

15. $2x^2+5x+2$.

HINT. Study the examples on pages 130–131.

16. $4a^2+7a+3$.

17. $3x^2+13x+12$.

18. $3x^2+17x+10$.

19. $4x^2+8x+3$.

20. $2b^2-5b+2$.

21. $6c^2+7c+2$.

22. $3x^2-11x+8$.

23. $10x^2-29x+10$.

24. $12x^2-11xy+2y^2$.

25. $2a^2+3a-2$.

26. $2a^2-a-15$.

27. $9a^2+3a-2$.

28. $12r^2+10r-12$.

29. $6a^4-11a^3b-7a^2b^2$.

30. $6x^3+10x^2y-4xy^2$.

31. $2x^2+5xy-12y^2$.

32. $6a^{2n}-7a^n+2$.

33. $3a^{2n}-10a^n-8$.

34. $10a^{2n}-a^n-3$.

138. Expressions reducible to the difference of squares. The type form is

$$a^4 + ka^2b^2 + b^4.$$

If k has such a value that the trinomial is not a perfect square, a trinomial of this type can often be written as the *difference of two squares.* Thus, if $k=1$, the addition and subtraction of a^2b^2 accomplishes this result.

EXAMPLES

1. Factor $a^4 + a^2b^2 + b^4$.

Solution.
$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2\,a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab). \end{aligned}$$

2. Factor $49\,h^4 + 34\,h^2k^2 + 25\,k^4$.

Solution. If $36\,h^2k^2$ is added, the expression becomes a perfect trinomial square. Adding and subtracting $36\,h^2k^2$, we have

$$\begin{aligned} 49\,h^4 + 34\,h^2k^2 + 25\,k^4 &= 49\,h^4 + 70\,h^2k^2 + 25\,k^4 - 36\,h^2k^2 \\ &= (7\,h^2 + 5\,k^2)^2 - (6\,hk)^2 \\ &= (7\,h^2 + 5\,k^2 + 6\,hk)(7\,h^2 + 5\,k^2 - 6\,hk). \end{aligned}$$

EXERCISES

Factor:

1. $x^4 + x^2y^2 + y^4$.
2. $c^4 + c^2d^2 + d^4$.
3. $a^4 + a^2b^4 + b^8$.
4. $a^4 + 3\,a^2b^2 + 4\,b^4$.
5. $m^4 + m^2 + 1$.
6. $x^4 + 5\,x^2 + 9$.
7. $c^4 + 4\,c^2 + 16$.
8. $25\,a^4 - 19\,a^2 + 1$.
9. $25\,x^4 - 11\,x^2 + 1$.
10. $4\,r^4 - 44\,r^2s^2 + 49\,s^4$.
11. $25\,x^4 - 19\,x^2 + 9$.
12. $9\,ax^8 - 28\,ax^4y^4 + 4\,ay^8$.
13. $4\,a^8x + 3\,a^4b^4x + 9\,b^8x$.
14. $4\,a^4 + 1$.

HINT. $4\,a^4 + 1 =$
$4\,a^4 + 4\,a^2 + 1 - 4\,a^2$.

15. $c^4 + 4\,d^4$.
16. $64\,a^4x^2 + x^2$.
17. $4\,a^{4n} + b^{4n}$.
18. $x^{4m} + 4\,y^{4n}$.

139. A binomial the sum or the difference of two cubes. Study the examples on page 133.

EXERCISES

Factor:

1. $a^3 + d^3$.
2. $a^3 + 2^3$.
3. $d^3 + 8$.
4. $x^3 + 125$.
5. $x^3 - y^3$.
6. $x^3 - 2^3$.
7. $m^3 - 64$.
8. $m^3 - (2n)^3$.
9. $8x^3 - y^3$.
10. $125 - x^3$.
11. $a^3 + (b^2)^3$.
12. $c^3 + d^6$.
13. $m^3 - n^9$.
14. $(2a)^3 - (3b)^3$.
15. $125x^3 + 8y^3$.
16. $(a + b)^3 + c^3$.
17. $m^3 - n^3 - m + n$.
18. $x^3 - 8y^3 + x - 2y$.
19. $r^3 - 8s^3 + r - 2s$.
20. $a^{3n} + b^{3n}$.
21. $a^{3m} - b^{3n}$.

22. Describe the process of factoring the sum of two cubes; the difference of two cubes.

140. The Remainder Theorem. *If any rational integral expression in x be divided by $x - n$, the remainder is the same as the original expression with n substituted for x.* Although a proof of this fact is too difficult for inclusion here, its truth is illustrated in the following:

EXAMPLE

Divide $x^2 - 5x + 6$ by $x - n$.

Solution.

$$\begin{array}{l|l}
x^2 - \ \ 5x + 6 & x - n \\
\cline{2-2}
\underline{x^2 - \ \ nx \qquad\ } & \underline{x + (n - 5)} \\
(n - 5)x + 6 & \\
\underline{(n - 5)x \qquad - n^2 + 5n} & \\
\qquad\qquad\quad n^2 - 5n + 6 = \text{Remainder} &
\end{array}$$

Here the remainder $n^2 - 5n + 6$ is the same as $x^2 - 5x + 6$, the given expression, when n is substituted for x.

EXERCISES

1. Divide $x^2 + bx + c$ by $x - n$ and show that the remainder is $n^2 + bn + c$.

2. Divide $x^2 + bx + c$ by $x - a$ and find the remainder.

3. Divide $x^3 + ax^2 + bx + c$ by $x - n$ and find the remainder.

4. In $(x^3 + x^2 - 5x + 3) \div (x - 2)$ find the remainder (*a*) by division, (*b*) by the Remainder Theorem.

Solution (*b*). $2^3 + 2^2 - 5 \cdot 2 + 3 = 5$.

5. In $(x^3 - x + 5) \div (x - 3)$ find the remainder (*a*) by division, (*b*) by the Remainder Theorem.

By use of the Remainder Theorem find the remainders in the following:

6. $(x^3 + x^2 - 5x + 8) \div (x - 3)$.

7. $(x^3 - 3x - 15) \div (x + 4)$.

8. $(x^3 - 2x^2 - 100) \div (x - 5)$.

9. $(x^3 - 2x^2 - 2x - 3) \div (x - 3)$.

10. $(x^4 - 2x^3 + x - 2) \div (x - 2)$.

141. Factor Theorem. By substituting 2 for x in x^2-5x+6 we obtain $4 - 10 + 6$, or 0. Hence $x - 2$ is an exact divisor (or factor) of $x^2 - 5x + 6$. Again, if 3 is substituted for x in $x^2 - 5x + 6$, the expression equals zero. Hence $x - 3$ is a factor of $x^2 - 5x + 6$. The foregoing examples illustrate the

Theorem. *If any rational integral expression in* x *becomes zero when a number* n *is substituted for* x*, then* $x - n$ *is a factor of the expression.*

The Factor Theorem may be used to factor some of the preceding exercises and, in addition, many others which are very difficult to factor by previous methods.

When searching for the values of x which will make an expression zero, only integral divisors of the last term of the expression (arranged according to the descending powers of x) need be tried, for the last term of the factor must be an integral divisor of the last term of the expression.

NOTE. By means of the Factor Theorem we are able to solve cubic and higher equations when the roots are integers. The solution of the general cubic equation is one of the famous problems of mathematics and one which is accompanied by many interesting applications. This problem was first solved by the Italian, Tartaglia, about 1530, but was published by Cardan, to whom Tartaglia explained his solution on the pledge that he would not divulge it. For many years the credit for the discovery was given to Cardan, and to this day it is usually called Cardan's Solution.

EXAMPLE

Factor $x^3 + 2x - 3$.

Solution. If $x - n$ is a factor of $x^3 + 2x - 3$, then n must be an integral divisor of 3. Now the factors of -3 are $1, -1, 3$, and -3. If 1 is put for x, then $x^3 + 2x - 3$ equals zero, hence $x - 1$ is a factor of $x^3 + 2x - 3$. Dividing $x^3 + 2x - 3$ by $x - 1$, we obtain the quotient $x^2 + x + 3$. Since $x^2 + x + 3$ is prime, the factors of $x^3 + 2x - 3$ are $x - 1$ and $x^2 + x + 3$.

ORAL EXERCISES

1. Is $x - 1$ a factor of $x^3 + 3x - 4$?
2. Is $x - 2$ a factor of $2x^3 + x^2 - 20$?
3. Is $a - 2$ a factor of $a^3 - 3a + 2$?
4. Is $x - 1$ a factor of $x^3 + 3x^2 - 4$?
5. Is $r + 1$ a factor of $r^3 - 4r^2 - 4r + 1$?
6. Is $r - 3$ a factor of $2r^3 - r^2 + 5$?
7. Is $s + 1$ a factor of $3s^3 - 5s^2 + 8$?
8. Is $k - 3$ a factor of $2k^3 - 5k^2 - 9$?

EXERCISES

Factor:

1. $x^3 + x - 2$.
2. $x^3 + 2x + 3$.
3. $a^3 + a^2 - 36$.
4. $x^3 + x - 10$.
5. $d^3 + d^2 - 12$.
6. $x^3 - x^2 + 4x - 4$.
7. $x^3 - 2x^2 - 5x + 6$.
8. $x^3 - x^2 - 4$.
9. $x^4 - 7x^2 - 6x$.
10. $y^3 - y^2 - 9y + 9$.
11. $2x^3 - 2x^2 - x - 6$.
12. $x^3 - 7x^2 + 4x + 12$.
13. $a^4 - 6a^3 + 11a^2 - 6a$.
14. Describe the process of factoring the above.

142. The sum or difference of two like powers. The type form is

$$a^n \pm b^n.$$

The cases in which $a^n \pm b^n$ is divisible by $a + b$ or $a - b$ can be determined by the Factor Theorem.

Thus in $a^n - b^n$, n being either an odd or an even integer, substitute b for a. Then $a^n - b^n$ becomes $b^n - b^n = 0$. Therefore $a - b$ is always a factor of $a^n - b^n$.

In $a^n - b^n$, n being even, put $-b$ for a. Then $a^n - b^n$ becomes $b^n - b^n = 0$, since $(-b)^n$ is *positive* when n is *even*. Therefore, when n is even, $a + b$ as well as $a - b$ is an exact divisor of $a^n - b^n$.

In $a^n + b^n$, n being even, put either $+b$ or $-b$ for a. Then $a^n + b^n$ becomes $b^n + b^n$, which is not zero. Therefore $a^n + b^n$ is never divisible by $a + b$ or $a - b$ when n is even.

In $a^n + b^n$, n being odd, put $-b$ for a. Then $a^n + b^n$ becomes $(-b)^n + b^n = 0$, since $(-b)^n$ is *negative* when n is *odd*. Therefore, when n is odd, $a + b$ is a divisor of $a^n + b^n$.

Summing up:

I. $a^n - b^n$ is *always* divisible by $a - b$.

II. $a^n - b^n$, when n is *even*, is divisible both by $a + b$ and by $a - b$.

III. $a^n + b^n$ is *never* divisible by $a - b$.

IV. $a^n + b^n$, when n is *odd*, is divisible by $a + b$.

ORAL EXERCISES

For each of the following, state a binomial factor:

1. $x^3 - y^3$.
2. $x^3 - 5^3$.
3. $27 - a^3$.
4. $x^5 - 2^5$.
5. $32x^5 - y^5$.
6. $x^7 - y^7$.
7. $x^7 - 2^7$.
8. $2^7x^7 - y^7$.
9. $a^4 - c^4$.
10. $a^8 - b^8$.
11. $m^3 + 2^3$.
12. $a^3 + 8$.
13. $x^5 + y^5$.
14. $r^5 + 2^5$.
15. $a^5 + 32$.
16. $1 - r^7$.
17. $1 - r^n$.
18. $1 + r^7$.
19. $1 + r^n$.
20. $5^3 + 1$.
21. $8^3 - 1$.
22. $10^3 - 1$.
23. Is $10^7 + 1$ divisible by 11?
24. Is $10^9 - 1$ divisible by 9?

EXAMPLE

Factor $x^5 + y^5$.

Solution. By division,

$$\frac{x^5 + y^5}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4.$$

Hence $$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).$$

Note that the signs of the second factor are alternately plus and minus. Also note the order in which the exponents occur.

EXERCISES

Factor:

1. $x^5 + z^5$.
2. $x^5 + 1$.
3. $a^5 + 2^5$.
4. $x^5 + 32$.
5. $(a^2)^5 + (b^2)^5$.
6. $x^5 - z^5$.

HINT. Find the second factor by division and observe the signs of the terms and the order in which the exponents occur.

7. $a^5 - 2^5$.
8. $a^5 - 32x^5$.
9. $(2x)^5 - 243y^5$.
10. $a^7 - x^7$.
11. $1 - r^7$.
12. $x^7 - 128$.
13. $x^{10} + y^{15}$.
14. $a^{10} + 32x^{15}$.
15. $x^7 + a^7$.
16. $1 + r^7$.
17. $128x^7 + 1$.
18. $1 - r^n$.

HINT. Write only the first five terms and the last term of the polynomial factor.

19. $1 + r^n$.
20. $x^6 - y^6$.

HINT. Factor first as the difference of two squares.

21. $x^8 - y^8$.
22. $a^{10} - b^{10}$.
23. $a^{12} - b^{12}$.

143. General directions for factoring. The following suggestions will prove helpful in factoring:

I. First look for a common monomial factor, and if there is one (other than 1), separate the expression into its greatest monomial factor and the corresponding polynomial factor.

II. Then from the form of the polynomial factor determine with which of the following types it should be classed, and use the methods of factoring applicable to that type.

1. $ax + ay + bx + by$.
2. $a^2 \pm 2ab + b^2$.
3. $a^2 - b^2$.
4. $x^2 + bx + c$.
5. $ax^2 + bx + c$.
6. $a^4 + ka^2b^2 + b^4$.
7. $a^3 \pm b^3$.
8. $a^n \pm b^n$.

III. Proceed again as in II with each polynomial factor obtained, until the original expression has been separated into its prime factors.

IV. If the preceding steps fail, try the Factor Theorem.

REVIEW EXERCISES

Factor:

1. $6x^4 + 2x^3 + 2x^2$.
2. $5a^3 + 2a^2 - 15a - 6$.
3. $2c^3d - 8cd^3$.
4. $3m^3 - 3m^2 - 18m$.
5. $2x^2 + 3ax + a^2$.
6. $x^4 - 7x^2y^2 + 9y^4$.
7. $a^4c - ac^4$.
8. $2x^7y - 2xy^7$.
9. $x^5 - 2x^4 - 9x^2$.
10. $18r^3 - 24r^2s + 8rs^2$.
11. $45x^3y - 20xy^3$.
12. $2h^3k + 4h^2k^2 - 30hk^3$.
13. $x^4 + 7x^2 + 16$.
14. $a^4d - 8ad^4$.
15. $2a^7 + 64a^2$.
16. $m^3n + mn^3$.
17. $x^3 + 4x - 5$.
18. $ax^2 - 4a + 3x^2 - 12$.
19. $a^2c^2 - c^2 - a^2 + 1$.
20. $x^4 - 13x^2 + 36$.
21. $a^5 - a^2b^3 - a^3b^2 + b^5$.
22. $x^{12} - x^2$.

23. $x^3 - 3x^2 - 4x + 12.$

24. $(a + x)^2 + 10(a + x) + 25.$

25. $a^5 + 27a^2.$

26. $(x + y)^2 - 6(x + y)z + 9z^2.$

27. $(a - x)^2 - 16(m - n)^2.$

28. $a^4 - 11a^2 + 1.$

29. $a^4 + ab^{15}.$

30. $4(a - b)^2 - 12(a - b)c + 9c^2.$

31. $a^2 - 10ab + 25b^2 - c^2.$

32. $x^2 - 2x(a - b) - 35(a - b)^2.$

33. $x^3 - 8x^2 + 17x - 10.$

34. $x^4 - x^2y^2 + x^3 - x^2y.$

35. $a^2 - b^2 + (a - b)^2.$

36. $c^2 - 2cd + d^2 - 2(c - d) - 35.$

37. $2(a + b)^2 + 3c(a + b) - 2c^2.$

38. $a^3 + b^3 + 3a^2b + 3ab^2.$

39. $32x^8 - x^3y^{10}.$

40. $c^2 + 4d^2 - x^2 - 4cd.$

41. $(a - 2x)c^2 + (2x - a)d^2.$

42. $6a^4 + 3a^3 - 3a^2.$

43. $16a^4 + 7a^2 + 1.$

44. $x^6y^6 - 64.$

45. $a^7 - a^3b^6 + a^2b - a^3.$

46. $a^4 + 8ab^3 + a^2 + 2ab.$

47. $9x^2 - 4y^2 - 3x - 2y.$

48. $x^2 - 5(2x - 5).$

49. $3a^2 - 14a(b - c) + 8(b - c)^2.$

50. $a^4x^4 + 4.$

51. $x^8y^6 + x^2z^6.$

52. $x^3 - 10x - 3.$

53. $a^8 - 5a^4 + 4.$

54. $x - 1 + x^5 - x^3.$

55. $x^3 + x - y - y^3.$

56. $x^2 + 2xy + y^2 - 25a^2 - 10a - 1.$

57. $a^4 - b^4 - 2ab(a^2 - b^2).$

58. $2x^3 \pm 3x^2 + x.$

59. $m^7 - 8m - 7m^4.$

60. $2x^4 - 10x^2 + 4x.$

61. $a^3 + 1 + 3a^2 + 3a.$

62. $12x^2 - 8x + x^4 - 6x^3.$

63. $a^nx^2 + 2a^nx + a^n.$

64. $mr - ms - nr + ns.$

65. $h^{2m} - 2h^mk^n + k^{2n}.$

66. $a^{2m} - b^{2n}.$

67. $x^{2n} + (r + s)x^n + rs.$

68. $hrx^{2n} + hrx^n + hsx^n + hs.$

69. $a^{4m} + a^{2m}b^{2n} + b^{4n}.$

70. $a^{3m} + b^{3n}.$

71. $a^{3m} - b^{3n}.$

72. $a^{5m} + b^{5n}.$

73. $a^{6m} - b^{6n}.$

74. Solve for x, $ax + bx - 3a = 3b.$

Solution. Rewriting, $ax + bx = 3a + 3b.$

Factoring in each member, $x(a + b) = 3(a + b).$

Dividing each member by $a + b$, $x = 3.$

75. Solve for x, $cx + dx = c^2 + cd$.

76. Solve for m, $ma - mc + bc = ab$.

77. Solve for y, $5ay - 3by - 5a^2 + 3ab = 0$.

78. Solve for z, $az - 3ad = bz - 3bd$.

79. Solve for x, $cx - 2dx = c^2 - 4d^2$.

80. Solve for r, $r(2a - 7c) = 4a^2 - 28ac + 49c^2$.

81. Solve for m, $m(3a - c) - 9ad = 2ec - 6ae - 3cd$.

82. Solve for x, $ax - 3bx - a^2 = 3b^2 - 4ab$.

83. Solve for y, $y + 21d^2 + 4d = 7dy + 1$.

84. Solve for x, $d(1 - 3a) + x + 3ac = c + 3ax$.

85. Solve for z, $az + ae - 2ec + 2cd = 2cz + ad$.

86. Solve for x, $5x - 2cx - a(5 - 2c) = 5a + 5 - 2c - 2ac$.

87. Solve for x, $a^2x - ax + x - a^3 - 1 = 0$.

88. Solve for x, $c^3 - c^2x - 2cdx - 8d^3 = 4d^2x$.

89. Solve for m, $m(a - 2)(a^2 + 4) = a^4 - 16$.

90. Solve for s, $16s - 8as + 4a^2s - 2a^3s + a^4s - 32 = a^5$.

144. The solution of equations by factoring. Study the principle and examples on pages 139–140.

EXERCISES

Solve by factoring:

1. $x^2 - 4 = 0$.

2. $x^2 - 5x = 0$.

3. $x^2 - 4a^2 = 0$.

4. $y^3 = 64y$.

5. $x^2 + 5 = 6x$.

6. $2x^2 + x = 6$.

7. $m^2 - am - 2a^2 = 0$.

8. $y^3 - y + 2 = 2y^2$.

9. $x^2 - 9a^2 - x + 3a = 0$.

10. $x^3 - a^2x + 2ax^2 - 2a^3 = 0$.

11. $y^3 - 7y - 6 = 0$.

12. $y^4 - 13y^2 + 36 = 0$.

13. $x^5 - 5x^3 = -4x$.

14. $z^3 + 12z - 6z^2 - 8 = 0$.

15. $3x^3 + 72x + 33x^2 = 0$.

16. $8x^2 + 71x^3 - 9x^4 = 0$.

17. $(2x - 3)^2 - (5x + 6)^2 = 0$.

18. $(x - 3)^2 - 2(x - 3) = 8$.

19. $x^3 + 50 - 25x - 2x^2 = 0.$

20. $x^3 - ax^2 - 4a^2x + 4a^3 = 0.$

21. $y^3 + 3y^2 + 3y + 1 = 0.$

22. $acx^2 + bcx + adx + bd = 0.$

23. $x^2 - 7x - 8 = x + 1.$

24. $3x^2 - 11x + 6 = 3x - 2.$

25. $6x^2 + 7x - 3 = 3x - 1.$

26. State in general terms the process of solving the above equations.

145. The highest common factor. The **highest common factor** (H.C.F.) of two or more monomials or polynomials is the expression of highest degree, with the greatest numerical coefficient, which is an exact divisor of each.

Thus the H.C.F. of $28a^2b^3$ and $42a^3b^2$ is $14a^2b^2$. The H.C.F. of $x^3 - 4x$ and $x^3 - 5x^2 + 6x$ is $x(x - 2)$, or $x^2 - 2x$. Again, the H.C.F. of $(a + b)(a - b)$ and $a(b - a)$ is $a - b$ or $b - a$.

EXAMPLE

Find the H.C.F. of $9x^4 - 36x^2$ and $3x^7 - 12x^6 + 12x^5$.

Solution. Factoring, we have

$$9x^4 - 36x^2 = 3^2x^2(x + 2)(x - 2),$$

$$3x^7 - 12x^6 + 12x^5 = 3x^5(x - 2)^2.$$

Therefore the H.C.F. is $3x^2(x - 2)$, which equals $3x^3 - 6x^2$.

The method used in the preceding solutions for finding the H.C.F. of two or more monomials or polynomials is stated in the

Rule. *Separate each expression into its prime factors. Then find the product of such factors as occur in each expression, using each prime factor the least number of times it occurs in any one expression.*

If two or more polynomials have no common factor other than 1, then 1 is their H.C.F., and the polynomials are said to be prime to each other.

EXERCISES

Find the H.C.F. of the following:

1. $12, 18, 24.$
2. $15, 25, 40.$
3. $12a^2, 30a^5, 36a^4.$
4. $28a^2b^4, 42ab^5, 70a^2b^3.$
5. $66c^4x, 132c^2x^3, 165c^4x^2.$
6. $a^2 + 2ab + b^2, a^2 - b^2.$
7. $3a^2 - 3b^2, 9(a - b)^2, 3a^3 - 3b^3.$
8. $ax^2 - 2axy + ay^2, a^2x^2 - a^2y^2, 2ax^3 - 2ay^3.$
9. $2a^2m^2 - 2a^2n^2, 4am^2 - 12amn + 8an^2, 10am - 10an.$
10. $25x^3 - 25x^2y, 10x^3 - 20x^2y - 30xy^2, 5x^4 - 5xy^3.$
11. $24a^6 - 6a^4b^2, 48a^6 + 24a^5b, 48a^5 - 48a^4b - 36a^3b^2.$
12. $5x^7 - 160x^2, 15x^5 - 60x^3, 25x^7 - 200x^4.$
13. $18a^4 - 2a^2b^2, 12a^6 - 8a^5b - 4a^4b^2, 30a^4b^2 + 10a^3b^3.$
14. $a^5 - 3a^4b + 2a^3b^2,\ a^5 - 2a^4b - a^3b^2 + 2a^2b^3,$
    $a^7 - 5a^5b^2 + 4a^3b^4.$

NOTE. The most famous, and in some respects the most perfect, treatise on elementary mathematics ever written is Euclid's "Elements." About one third of the material of the thirteen books treats topics which to-day would be considered arithmetical in character. In appearance and language, however, they are all geometrical, for Euclid represents quantities not by numerals, as we do in arithmetic, or by letters, as we do in algebra, but by lines. Book VII contains the earliest statement of a general method for finding the G.C.D. of two numbers. This method, though never necessary in elementary mathematical work, is so perfect and beautiful from a scientific point of view that until recently it remained in elementary treatises on algebra and arithmetic by force of tradition. It is a great tribute to Euclid's genius that he was able to devise so perfect a method for the process that all the efforts of two thousand years have been unable to improve it essentially. It is of fundamental importance in advanced portions of algebra.

FRACTIONS

146. Reduction of fractions to lowest terms. Study the principle and illustrations on page 148 along with the examples on page 150.

EXERCISES

Reduce to lowest terms:

1. $\dfrac{2a^3b^2}{10a^2b^3}$.
3. $\dfrac{x^2-9}{2x^2+6x}$.
5. $\dfrac{48x^4-6xy^3}{32x^4-32x^3y+8x^2y^2}$.
2. $\dfrac{15xy^3}{245x^3y}$.
4. $\dfrac{5x^2+5xy}{25x^4-25x^2y^2}$.
6. $\dfrac{3x^3-375}{2x^2+10x+50}$.
7. $\dfrac{2c^2+2cd-4d^2}{5c^2-5d^2}$.
10. $\dfrac{4x^2+6x-40}{4x-15a+6ax-10}$.
8. $\dfrac{2c^6-64cd^5}{16c^4-40c^3d+16c^2d^2}$.
11. $\dfrac{5x^4-40x}{3x^5-96}$.
9. $\dfrac{ad-3ax+2cd-6cx}{ad-3ax-cd+3cx}$.
12. $\dfrac{12a^3+10a^2-12a}{4+9a^2-12a}$.
13. $\dfrac{32x^2-x^7}{32+16x+4x^3+8x^2+2x^4}$.
14. $\dfrac{(\frac{2}{3})^2-1}{\frac{2}{3}+1}$.
15. $\dfrac{(\frac{3}{5})^3-1}{\frac{3}{5}-1}$.
16. $\dfrac{(\frac{3}{2})^2-(\frac{3}{2})(\frac{2}{5})+(\frac{2}{5})^2}{(\frac{3}{2})^3+(\frac{2}{5})^3}$.

147. Reduction to equivalent fractions. Study the example on page 152, the solution on page 153, the examples on page 154, the rule on page 155, and the solutions on page 156.

EXERCISES

Change to respectively equivalent fractions having the L.C.D.

1. $\dfrac{2}{3x}, \dfrac{5}{2x^2}$.
4. $\dfrac{2x-1}{x^2-6x+9}, \dfrac{-x+5}{-2x+6}$.
2. $\dfrac{2x}{x-2}, \dfrac{3}{x+2}$.
5. $\dfrac{x+1}{x^2-5x+6}, \dfrac{-x}{-x^2+4}$.
3. $\dfrac{3}{a^2-9}, \dfrac{2a}{a-3}$.
6. $\dfrac{2a}{a+2}, \dfrac{a+3}{3a+1}, \dfrac{2a^2+5}{3a^2+7a+2}$.

7. $\frac{a}{2x}, \frac{2a}{3x}, \frac{5a}{x^2+2x}$.

8. $\frac{2x+1}{x^2-2x+4}, \frac{5}{x^3+8}, \frac{1}{x+2}$.

9. $\frac{2x}{x^3-8}, \frac{5x}{2-x}, \frac{4}{4-x^2}, \frac{3x+2}{x^3+2x^2+4x}$.

10. $\frac{2x-9}{x^5-32}, \frac{-2}{-x^3+2x^2}, \frac{x-2}{x^5+2x^4+4x^3+8x^2+16x}$.

148. Addition and subtraction of fractions. To find the **algebraic** sum of two or more fractions in their lowest terms we proceed as in the example on page 159 and in the solution on page 161.

EXERCISES

Find the algebraic sum of:

1. $\frac{2x}{5} + \frac{3x}{10} - \frac{2x}{15}$.

2. $\frac{2c}{3a} - \frac{5c^2}{6a^2} + \frac{c}{9a}$.

3. $\frac{2rs^3+r}{r^2s} - \frac{r-6s}{3rs^2} - \frac{s^2}{2r}$.

4. $\frac{5a}{a-3} - \frac{2}{3}$.

5. $\frac{c}{c-4} - \frac{2c}{c+3}$.

6. $\frac{a}{a-5} + \frac{2a}{a+4} - \frac{1}{2}$.

7. $\frac{2x+1}{x^2-9} - \frac{4}{x-3}$.

8. $\frac{a-3}{a^2-4} - \frac{a-3}{2-a} + \frac{5}{a+2}$.

9. $\frac{2c-3}{c(c-2)} - \frac{c^2-7}{8-c^3}$.

10. $\frac{m}{m-1} - \frac{2}{m} - \frac{5m-2}{m^2-1}$.

11. $2 + \frac{5x}{x-7}$.

HINT. $2 + \frac{5x}{x-7} = \frac{2}{1} + \frac{5x}{x-7}$.

12. $x + 3 - \frac{x^2}{x-3}$.

13. $x^2 + 2x + 4 - \frac{x^3+8}{x-2}$.

14. $a - \frac{a-6}{2a-3} - 2$.

15. $c^2 - \frac{c^4+d^4}{c^2+cd+d^2} - cd + d^2$.

16. $\dfrac{x+2}{2x-1}-\dfrac{x}{1-x}-\dfrac{3x^2+2x-2}{2x^2-3x+1}.$

17. $\dfrac{a-3}{a^2-2a-3}-\dfrac{2-a}{a^2-3a+2}-\dfrac{1}{1-a^2}.$

18. $\dfrac{b+a}{b-a}-2\left(\dfrac{b}{a}-\dfrac{b}{a-b}\right).$

19. $b-2a\left(1-\dfrac{2c-b}{c-b}\right).$

20. $\dfrac{x^2-(m-n)^2}{(x+n)^2-m^2}+\dfrac{m^2-(x-n)^2}{(x+m)^2-n^2}+\dfrac{n^2-(x-m)^2}{(m+n)^2-x^2}.$

21. Describe briefly the process of adding two or more fractions.

149. Multiplication and division of fractions. The method of multiplying fractions is illustrated in the example on page 167 and in the solution on page 169. The method to be followed in the division of fractions is shown in the illustrations on page 170.

EXERCISES

Perform the indicated operations:

1. $\dfrac{6a}{25x^2y}\cdot\dfrac{10xy^2}{4a^2x}.$

2. $5c\cdot\dfrac{3d}{10cd^2}\cdot\dfrac{4c}{9d}.$

3. $4\frac{1}{2}\cdot\dfrac{2a^2}{3x^3}\cdot\dfrac{x^2}{15a}.$

4. $10a\cdot\dfrac{7x}{2a^4}\cdot\dfrac{a}{21x^3}.$

5. $\dfrac{a^2-4}{2a^2}\cdot\dfrac{4a}{2a-4}.$

6. $\dfrac{a^2-1}{2a}\cdot\dfrac{6a^3}{a^2-2a+1}\cdot\dfrac{5a-5}{3(1-a)}.$

7. $\left(2a+\dfrac{1}{a}\right)\left(\dfrac{a^2}{4a^4-1}\right)\left(a-\dfrac{1}{2a}\right).$

8. $\frac{35}{42}\div\frac{25}{30}.$

9. $\dfrac{14x}{15ax^2}\div\dfrac{84a}{60a^2x^3}.$

10. $\dfrac{120c^2d}{42c^3x}\div\dfrac{100cd^3}{147c^2x^2}.$

11. Describe briefly the method used in multiplying two or more fractions; in dividing one fraction by another.

12. How is one mixed number multiplied by another?

13. How is one mixed number divided by another?

14. $\frac{24x^2}{99xy^2} \cdot \frac{8y^2z}{22x^2y} \div \frac{48x^3z}{121x^2y^3}$. **16.** $\frac{22m^2n^2}{35x^3} \div \frac{52m^5n}{125x^2} \div \frac{275n^2x}{39m^3x}$.

15. $\frac{15rs^2}{14s^2t} \div \frac{300rt^3}{98r^2s} \cdot \frac{160st^4}{28r^2s^3}$. **17.** $\frac{5x^2}{x-y} \cdot \frac{(x-y)^2}{20x^2} \div \frac{1}{x(x-y)}$.

18. $\frac{x+3}{x^2-4} \cdot \frac{x^2-4x+4}{2x+6} \div \frac{x^2-2x}{x^2+2x}$.

19. $\frac{x^2+2x+4}{x^2-2x+4} \div \frac{x^3-8}{x^2+2x} \div \frac{x^2-2x}{x^3+8}$.

20. $\frac{2x^2+9x+9}{2x^2-9x+9} \div \frac{x^2-9}{4x^3-9x} \cdot \frac{x^2-6x+9}{2x^2-3x}$.

21. $\frac{x^3-6x^2+11x-6}{x^4-1} \div \frac{x^2-5x+6}{x^2+2x+1} \cdot \frac{x^2+1}{x+1}$.

22. $\left(\frac{2x}{3}\right)^2\left(\frac{5}{4x}\right)^3\left(\frac{6x}{25}\right)^2$. **23.** $\left(\frac{3a}{2b}\right)^3 \div \left(\frac{3a^2x}{5b^2}\right) \div \frac{(10x)^3}{64}$.

24. $\left(\frac{2a}{5x^2}\right)^3 - 3\left(\frac{2a}{5x^2}\right)^2\left(\frac{5x^3}{4a^2}\right) + 3\left(\frac{2a}{5x^2}\right)\left(\frac{5x^3}{4a^2}\right)^2$.

25. $\left(\frac{a^2}{2b}\right)^6 + 6\left(\frac{a^2}{2b}\right)^5\left(\frac{2b}{a}\right) + 15\left(\frac{a^2}{2b}\right)^4\left(\frac{2b}{a}\right)^2 + 30\left(\frac{a^2}{2b}\right)^3\left(\frac{2b}{a}\right)^3$.

26. $\frac{1-a^3}{1+a^2} \div \left(1 - \frac{2a^6}{a^6+1}\right) \div \left(\frac{1-a^2+a^4}{a^3+1}\right)$.

27. $\frac{a}{a^2-4} \div \left(\frac{a}{3a^2+5a+2}\right)\left(3a - \frac{2}{a} - 5\right)$.

28. $\dfrac{x+1+\dfrac{4}{x+3}}{x-5+\dfrac{12}{x+3}}$.

HINT. An expression of this form is called a complex fraction. It is simply another way of writing

$$\left(x+1+\frac{4}{x+3}\right) \div \left(x-5+\frac{12}{x+3}\right).$$

29. $\dfrac{2+\dfrac{x+6}{x-2}}{3+\dfrac{5}{x-2}}$.

30. $\dfrac{x-1-\dfrac{3}{x}}{x+1+\dfrac{x}{x-3}}$.

31. $\dfrac{\dfrac{a}{b}+1+\dfrac{b}{a}}{\dfrac{b^2}{a^2}+1+\dfrac{a^2}{b^2}}$.

32. $\dfrac{\dfrac{a}{a+1}+\dfrac{1-a}{a}}{\dfrac{a}{a+1}-\dfrac{1-a}{a}}$.

33. $\dfrac{\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}}{1+\dfrac{2a^2b^2+2b^4}{a^4-b^4}}$.

34. $\left(\dfrac{x+y}{x-y}+2\right)\div\left(\dfrac{x+y}{x-y}-2\right)$.

35. $\dfrac{\dfrac{2}{b}-\dfrac{1}{a+b}+\dfrac{1}{a-b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$.

36. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{\dfrac{b}{a+b}}{1+\dfrac{2b-a}{a-b}}$.

37. $\dfrac{2}{\dfrac{1}{b^2}\left(\dfrac{1}{c}-\dfrac{b}{2a}\right)}-b^2c\left(1-\dfrac{bc}{a-\dfrac{bc}{2}}\right)$.

38. $\dfrac{2b}{\dfrac{c}{2a}-2}-\dfrac{b}{ac}\left(\dfrac{4ab}{\dfrac{4b}{c}-\dfrac{b}{a}}-\dfrac{a}{\dfrac{1}{c}-\dfrac{1}{b}}\right)$.

39. What value has $\dfrac{a^3-b^3}{a^2+b^2}$ when $a=x-\dfrac{1}{x}$ and $b=x+\dfrac{1}{x}$?

40. $1-\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{2}}}$.

41. $1-\dfrac{1}{1+\dfrac{2}{1-\dfrac{2}{3}}}$.

42. $1+\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{a}}}$.

150. Equations involving fractions. Equations involving fractions are solved as in the examples on pages 176–177 and in the hints on page 183.

EXERCISES

Solve the following for x and check as directed by the teacher:

1. $\dfrac{5x}{2}-2=2x$.

2. $\dfrac{2x}{5}+1=x-\dfrac{7}{5}$.

3. $\dfrac{11x^2}{9}-\dfrac{2x}{3}=x^2$.

4. $\dfrac{3x}{2}-\dfrac{16}{3}=x-\dfrac{25}{6}$.

5. $\frac{5x}{6}-\frac{21}{5}=\frac{5x-2}{10}.$

6. $\frac{2(x-4)}{3}-1=\frac{x+5}{3}.$

7. $\frac{2x+1}{4}-\frac{1-2x}{3}=9\frac{1}{4}.$

8. $\frac{3}{4x}+\frac{7}{16}=\frac{4}{3x}.$

9. $3(x+2)-4(2x-3)+2=0.$

10. $\frac{x+5}{x-2}=\frac{10}{3}.$

11. $\frac{8}{7x+3}=\frac{3}{3x+1}.$

12. $\frac{2x-7}{3x-4}+\frac{3}{2}=0.$

13. $(x+5)(x+1)-(x-3)(x-2)=10.$

14. $\frac{3}{2x-5}+\frac{1}{2x-1}=0.$

15. $\frac{2x-5}{2x+7}=\frac{3x-14}{3x-2}.$

16. $5x+3=\frac{(5x+7)(2x-3)}{2x}.$

17. $\frac{1}{x}-\frac{1-x}{6-x^2}=0.$

18. $\frac{x+3}{2x+1}=\frac{3x+4}{6x-2}.$

19. $\frac{x-3}{x+7}=1-\frac{5}{x+1}.$

20. $\frac{3(1-8x)}{5}-\frac{2x(1+8x)}{8x-1}=\frac{1-34x}{5}.$

21. $\frac{3}{2x+1}+\frac{2x+1}{1-2x}=1-\frac{8x^2}{4x^2-1}.$

22. $\frac{1+2x}{2x-1}-\frac{3x+1}{x+1}=\frac{8+3x-4x^2}{2x^2+x-1}.$

23. $\frac{x}{a}-a=\frac{x}{b}-b.$

24. $\frac{x+a}{a}-\frac{x+b}{b}=b-a.$

25. $\frac{x^2-x}{a-1}+\frac{x^2-x}{a+1}=2a(x-1)$

26. $\frac{x}{a+2}-\frac{x+16}{a-2}=4a.$

27. $\frac{2x+1}{x+3}+\frac{3x-7}{2-x}=\frac{9-3x-x^2}{x^2+x-6}.$

28. $\frac{x-a}{2x-a}-\frac{3x-c}{6x-c}=0.$

29. $11.3-\frac{5-2x}{.5}=2.3-(5-7x)+\frac{x+2}{.2}.$

30. $\frac{3x-1}{.25}+\frac{x-4}{.5}=3(3x-14).$

PROBLEMS

1. Separate the number 1010 into three parts such that the second will be $\frac{11}{6}$ of the first and the third will be $\frac{25}{9}$ of the first.

2. By what number must 352 be divided so as to give a partial quotient 15 and the remainder 7?

3. What number must be subtracted from both terms of the fraction $\frac{32}{39}$ to give a fraction equivalent to $\frac{2}{3}$?

4. Separate 96 into two parts such that 56 exceeds two thirds of the one by as much as the other exceeds 16.

5. A boy is 12 years old and his sister 8 years old. In how many years will the boy be $\frac{6}{5}$ as old as his sister?

6. The square of a certain number is 4 greater than two thirds of the product of the next two consecutive numbers. Find the number.

7. The length of a certain rectangle is $2\frac{1}{2}$ times the width. If it were 10 yards shorter and $1\frac{1}{2}$ yards wider, its area would be 1260 square feet less. Find the dimensions of the rectangle.

8. A can do a piece of work in 10 days and B in 12 days. How many days will they both require, working together?

HINT. See the solution of Exercise 20 on page 196.

9. A can do a piece of work in 10 days and B in 15 days. After they have worked together 5 days, how many days will A require to finish the work?

10. A tank has a supply pipe which fills it in 4 hours and a waste pipe which empties it in 6 hours. If the tank is empty and both pipes are opened, how much time must elapse before the tank is filled?

11. A tank has a supply pipe which fills it in 4 hours and two waste pipes which empty it in 6 and 8 hours respectively. If the tank is full and all three pipes are open, how much time will be required to empty the tank?

12. The diameter of the earth is $3\frac{2}{3}$ times that of the moon, and the difference of the two diameters is 5760 miles. Find each diameter in miles.

13. The diameter of Jupiter is $10\frac{10}{11}$ times the diameter of the earth, and the sum of their diameters is 94,320 miles. Find each diameter in miles.

14. A man who can row 4 miles per hour in still water rows up a stream the rate of whose current is $1\frac{1}{2}$ miles per hour. After rowing back he finds that the entire journey required 10 hours. Find the time required for the trip upstream.

15. A passenger train whose rate is 40 miles per hour leaves a certain station 2 hours and 45 minutes after a freight train. The passenger train overtakes the freight in 5 hours and 15 minutes. Find the rate of the freight train in miles per hour.

16. A man invests a part of $8000 at 5% and the remainder at 4%. If the yearly interest on the whole investment is $345, how much was invested at each rate?

17. Two thousand dollars of Mr. A's income was not taxed. All of his income over that amount was taxed 2%, and all above $10,000 was taxed 2% in addition. He paid a tax of $180. What was his income?

18. How much water must be added to a gallon of alcohol 95% pure so as to make a mixture 10% pure?

HINT. Let $w =$ the number of gallons of water to be added.

Then
$$\frac{\frac{95}{100}\cdot 1}{1+w}=\frac{10}{100}, \text{ etc.}$$

19. It is desired to mix coffee which sells for twenty-five cents per pound with coffee which sells for thirty-five cents per pound so as to obtain a 10-pound mixture which may be sold at the same profit for thirty-two cents per pound. In what ratio must the parts be taken from each grade?

20. Milk of a certain grade is known to test 20% butter fat. How much water must be added to 25 gallons of this milk to make a mixture 18% butter fat?

21. Gun metal of a certain grade is composed of 16% tin and 84% copper. How much tin must be added to 410 pounds of this gun metal to make a composition 18% tin?

HINTS. Since the composition is 16% tin, $\frac{16}{100} \cdot 410 =$ the number of pounds of tin in the first composition.

Let $x =$ the number of pounds of tin to be added.

Then $\frac{16 \cdot 410}{100} + x =$ the number of pounds of tin in the second composition,

and $410 + x =$ the number of pounds of both metals in the second composition.

Therefore $$\frac{\frac{16 \cdot 410}{100} + x}{410 + x} = \frac{18}{100}, \text{ etc.}$$

22. How many gallons of alcohol 90% pure must be mixed with 12 gallons of alcohol 96% pure to make a mixture 93% pure?

23. A certain lot of pig iron contains 93% pure iron. How much pure iron must be melted with 10 tons of pig iron to make iron 98% pure?

24. The arms of a lever are 5 feet and 6 feet in length respectively. Excluding the weight of the beam, what weight on the shorter arm will balance 70 pounds on the longer?

HINT. The products of the weights by their respective arms are equal.

25. The arms of a balanced lever are 7 feet and 12 feet respectively, the shorter arm carrying a load of 36 pounds. Find the load on the longer arm.

26. A beam 12 feet long supported at each end carries a load of 2 tons at a point 4 feet from one end. Find the load in tons on each support.

27. At what time between 3 and 4 o'clock will the hands of a clock be together?

Solution. The minute hand moves twelve times as fast as the hour hand. While the minute hand travels x spaces, the hour hand travels $\frac{x}{12}$ spaces. Hence $x - \frac{x}{12}$ equals the number of spaces gained by the minute hand in any given time x.

In the time from 3 o'clock until the hands are together, the minute hand must gain 15 minute spaces to overtake the hour hand.

Therefore $$x - \frac{x}{12} = 15.$$

Whence $$x = 16\tfrac{4}{11}.$$

Hence the hands are together $16\frac{4}{11}$ minutes after 3 o'clock.

28. At what time between 9 and 10 o'clock are the hands of a clock together?

29. At what times between 3 and 4 o'clock are the hands of a clock in a straight line?

30. At what time between 5 and 6 o'clock is the minute hand 10 minute spaces ahead of the hour hand? 10 minute spaces behind the hour hand?

31. At what times between 4 and 5 o'clock are the hands of a clock at right angles?

32. If the average height of n boys is x inches, what is the sum of their heights in yards?

33. If $3m + 5$ packages weigh p pounds, what is the weight of n of them?

34. If it takes a man t hours to do a piece of work, what portion of the work can he do in 1 hour? What portion of the work would n men do in 1 hour? What portion would k men do in h hours?

35. If it takes n men h hours to do a piece of work, how long will it take x men to do it?

36. A man buys bananas at d cents a dozen and sells them for b cents each. What does he gain on n dozen?

37. A man bought m articles for c cents per hundred. He sold them all for \$10. How many dollars did he lose?

38. If n yards of ribbon cost c cents, find the cost of x yards.

39. If y yards of ribbon cost x cents, how many yards can be bought for d dollars?

40. A man buys goods for b dollars and sells them for s dollars. What is his per cent of gain?

41. One man can do a piece of work in d days, another can do the same work in n days. How many days will it take both, working together?

42. If it takes h hours to mow m acres, how many days of 10 hours each will it take to mow n acres?

43. A train goes y yards in t seconds. If this equals m miles per hour, write an equation involving y, t, and m.

44. If n men can do a piece of work in d days, how many men would it be necessary to hire if the work had to be done in t days?

45. A transport plying between two ports is under fire for f feet of the way. If she steams k knots per hour, for how many minutes is she under fire? (1 knot = 6080 ft.)

LINEAR SYSTEMS

151. Graphical solution of a linear system. The construction of the graph of a single linear equation in two unknowns or of a linear system in two unknowns is explained on pages 206–209.

The *graphical solution* of a system in two unknowns consists in plotting the two equations to the same scale and on the same axes and obtaining from the graph the values of x and y at the point of intersection.

Since two straight lines can have only one point of intersection, two linear equations can have only one set of roots. In case the graphs of the equations of a system are parallel lines, the system has no set of roots, and a solution is impossible.

EXERCISES

Solve graphically:

1. $2x + y = 8,\quad x + 2y = 13.$

2. $x - y = 6,\quad 5x + 4y = -15.$

3. $2x + 4y = 20,\quad 2y - 2 = x.$

4. $x + y = 5,\quad y + 2 = 0.$

5. $x + y = 4,\quad x + 2y = 7.$

6. $x - y = 5,\quad 3x + 2y = 5.$

7. $2x - 4y = 9,\quad x - 2y = 8.$

8. In Exercises 5 and 6 will the values of the x- and y-distances of the point of intersection of the two lines, as obtained from the graph, satisfy the equation obtained by adding the two given equations?

9. Graph the equation $x - 2y = 8$. Then multiply both members by 3 and graph the resulting equation. Compare the two graphs. Then use -2 as a multiplier and graph the resulting equation. Compare the three graphs. What conclusion seems warranted?

10. What are the coördinates of the origin?

11. Is a graphical solution of a linear system ever impossible? Give an example.

12. What is the form of the equation of a line parallel to the x-axis? the y-axis?

152. Elimination. In order to find values of x and y which satisfy the equation

$$3x + 2y = 20, \tag{1}$$

when we know that

$$y = 2x + 3, \tag{2}$$

we may substitute for y in the first equation the value of y from the second, obtaining the single equation in x,

$$3x + 2(2x + 3) = 20;\ \text{whence}\ x = 2.$$

Substituting 2 for x in (2) we get $y=7$. The set of roots for equations (1) and (2) is $x=2$, $y=7$. The process by which we have obtained one equation containing one unknown from the two equations (1) and (2) each of which contains two unknowns illustrates one method of elimination.

In general, the process of deriving from a system of n equations a system of $n-1$ equations, containing one variable less than the original system, is called **elimination.**

For example, when $n=2$, if we have a system of two equations in two unknowns, the process of elimination leads to one equation in one unknown. Since we can always solve such an equation, it appears that we can solve a system of two equations in two unknowns whenever it is possible to eliminate one and only one of the unknowns. We shall see that only in certain exceptional cases is elimination impossible. This is either because more than one unknown is removed by the elimination or because the result of the attempt at elimination is not an equation.

Only two methods of solution will be considered — that involving **elimination by substitution** and that involving **elimination by addition or subtraction.**

153. Solution by substitution. The method of solving a system of two linear equations by substitution is explained in section 97.

EXERCISES

Solve by substitution:

1. $3x-8y=20$, $x-6y=0$.

2. $2x+5y=8$, $x-10y=9$.

3. $5=x+y$; $2(x+y)+3y=4$.

4. $16x+7=15y$, $4x+5y=0$.

5. $3x-2y=18$, $4y+3x=0$.

6. $\frac{8m-3n}{2}+6n=-9$, $4m-1=3n$.

154. Solution by addition or subtraction. The method of solving a system of two linear equations by addition or subtraction is explained in section 96.

EXERCISES

Solve the following systems by addition or subtraction:

1. $\begin{aligned} x + y &= 4, \\ x - y &= 2. \end{aligned}$

2. $\begin{aligned} x + 2y &= 3, \\ x + y &= 2. \end{aligned}$

3. $\begin{aligned} x - y &= 5, \\ 2x + y &= 4. \end{aligned}$

4. $\begin{aligned} 2x + 3y &= 8, \\ x - y &= 1. \end{aligned}$

5. $\begin{aligned} x + 3y &= 0, \\ 2x - 3y &= 9. \end{aligned}$

6. $\begin{aligned} 5x - 6y &= 7, \\ 4x - 3y &= 2. \end{aligned}$

155. Special cases. The equation $x + y = 10$ has as roots any set of two numbers whose sum is 10. If $x + y = 5$ is taken as the other equation of a system, one can see immediately that the two equations have no set of roots in common, since the sum of two numbers cannot be 10 and 5 at the same time.

A system of equations which has a common set of roots is called a **simultaneous** system.

A system of equations which does not have a common set of roots is called **inconsistent** or **incompatible.**

The attempt to solve an incompatible system results in getting rid not only of one but of both unknowns and leads to a statement in the form of an equation which is false.

Consider $\quad x + y = 10, \quad$ (1)

$x + y = 5. \quad$ (2)

(1) − (2), $\quad 0 = 5$, which is false.

If, on the other hand, the equation $x + y = 10$ is taken for one equation of a system and $2x + 2y = 20$ for the other, it appears that any set of numbers which satisfies one equation satisfies also the other, since if the sum of two numbers is 10, the sum of twice those numbers is 20, and any one of the countless sets of roots of one equation is a set of roots of the other. In fact, the second equation may be obtained from the first by multiplying each member by 2.

If one equation of a system can be obtained from one or more of the other equations of the system by application of one or more of the axioms(see page 39), it is called a **derived** or **dependent** equation. If it cannot be so obtained, it is called **independent**. Two equations one of which is dependent on the other constitute an **indeterminate system.**

Thus equations (1) and (2) in the example on page 220 are independent and together form a determinate system, while equation (5) is derived from them.

An attempt to eliminate one unknown from a system of two equations in two unknowns which are not independent results in getting rid not only of both unknowns but of the constant terms as well, so that only the identity $0 = 0$ remains.

Thus $$x + y = 10, \tag{1}$$

$$2x + 2y = 20. \tag{2}$$

(1) · 2, $$2x + 2y = 20. \tag{3}$$

(2) − (3), $$0 = 0.$$

EXERCISES

Solve the following systems:

1. $\dfrac{2x}{3} + y = -7,$
   $5x - 3y = 105.$

2. $\dfrac{3r}{4} - \dfrac{7}{2} = \dfrac{s}{12},$
   $r + 8 = -2s.$

3. $\dfrac{3y+1}{4} - \dfrac{z+22}{12} = 3,$
   $z - 2y = 1.$

4. $.5x + .7y = \frac{3}{5},$
   $.8x - .2y = 3\frac{3}{5}.$

5. $$\begin{aligned}&\frac{m+2n}{5}-\frac{2m-n}{10}=\frac{5}{2},\\&\frac{m+n}{4}-\frac{m-n}{7}=\frac{n}{2}.\end{aligned}$$

6. $$\begin{aligned}&\frac{1}{x}+\frac{1}{y}=-\frac{1}{6},\\&\frac{2}{x}-\frac{3}{y}=\frac{4}{3}.\end{aligned}$$

7. $$\begin{aligned}&\frac{2r+4s}{2r-s}=\frac{38}{3},\\&\frac{5}{r}-\frac{7}{s}=0.\end{aligned}$$

8. $$\begin{aligned}&\frac{3}{x-3}-\frac{1}{y}=0,\\&\frac{x-6y}{5}=\frac{2}{5}.\end{aligned}$$

9. $$\begin{aligned}&\frac{3(x+y)}{\frac{3}{5}}+\frac{x-y}{\frac{-11}{5}}=0,\\&2x+y=7.\end{aligned}$$

10. $$\begin{aligned}&\frac{3x+z}{z-x}+2=\frac{12}{x-z},\\&\frac{x+7}{z}+\frac{17+x}{5-z}=0.\end{aligned}$$

Solve for x and y:

11. $$\begin{aligned}&5x+4y=10a+4,\\&x-2ay=0.\end{aligned}$$

12. $$\begin{aligned}&2ax-3by=7,\\&5ax+7by=3.\end{aligned}$$

13. $$\begin{aligned}&\frac{x}{a}-\frac{y}{c}=6,\\&\frac{x}{2a}+\frac{y}{3c}=13.\end{aligned}$$

14. $$\begin{aligned}&\frac{a}{x}+\frac{3a}{y}=1,\\&\frac{3a}{x}+\frac{a}{y}=\frac{1}{2}.\end{aligned}$$

15. $$\begin{aligned}&\frac{x}{a}+\frac{y}{b}=\frac{a+b}{ab},\\&x-y=\frac{a^2-b^2}{ab}.\end{aligned}$$

16. $$\begin{aligned}&2ax-4by=7c,\\&3ax-6by=5c.\end{aligned}$$

156. Determinate systems in three or four unknowns. The method of obtaining the set of roots of a system in three unknowns is explained in section 100.

EXERCISES

Solve for x, y, and z and check the results:

1. $$\begin{aligned}&x+3y-5z=2,\\&2x-y-z=1,\\&3x+5y-7z=-10.\end{aligned}$$

2. $$\begin{aligned}&x-y+3z=0,\\&5x+2y+z=14,\\&2x+3y+4z=-14.\end{aligned}$$

$$\textbf{3.}\quad \begin{array}{l} x+2y+z=-1, \\ 2x-y+z=-20, \\ -x-y-5z=13. \end{array}$$

$$\textbf{4.}\quad \begin{array}{l} x+y+z=1, \\ x+y-z=2, \\ x-y+z=3. \end{array}$$

$$\textbf{5.}\quad \begin{array}{l} x+2y+z=1, \\ 2x+y-z=0, \\ x+2y+z=0. \end{array}$$

$$\textbf{6.}\quad \begin{array}{l} 2x-y+5z=0, \\ 8x+7y+z=38, \\ x-5y-z=7. \end{array}$$

$$\textbf{7.}\quad \begin{array}{l} 4x-3y=z, \\ z=x+y, \\ 2x=3y+1. \end{array}$$

$$\textbf{8.}\quad \begin{array}{l} x-2y=10, \\ 3y+4z=-1, \\ 5x-z=18. \end{array}$$

$$\textbf{9.}\quad \begin{array}{l} x+y=3a, \\ x+z=4a, \\ y+z=5a. \end{array}$$

$$\textbf{10.}\quad \begin{array}{l} .3x+.2y+.4z=1.9, \\ .02x=.1-.01y-.02z, \\ x+y+z=6. \end{array}$$

$$\textbf{11.}\quad \begin{array}{l} \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=3, \\ \frac{a}{x}-\frac{b}{y}+\frac{2c}{z}=2, \\ \frac{5a}{x}-\frac{2c}{z}=\frac{3b}{y}. \end{array}$$

$$\textbf{12.}\quad \begin{array}{l} \frac{x+y}{3}-\frac{z}{5}=\frac{2}{3}, \\ \frac{y+z}{4}-\frac{x}{2}=\frac{5}{12}, \\ 3x+2y=12-3z. \end{array}$$

$$\textbf{13.}\quad \begin{array}{l} 2x+y+z+w=1, \\ x-y-z+2w=4, \\ x+2y-z-w=0, \\ x-y+2z-w=1. \end{array}$$

$$\textbf{14.}\quad \begin{array}{l} x+y+z=1, \\ x-y-w=-1, \\ x-z-w=-5, \\ y-z+w=0. \end{array}$$

PROBLEMS

Express the conditions of the following problems by means of simultaneous systems, solve, and check the results:

1. The sum of two numbers is 109 and their difference is 49. Find the numbers.

2. Thirty-nine tons of material are to be moved by motor trucks and drays. It is found that the work can be done in a given time either by 10 trucks and 6 drays, or by 8 trucks and 10 drays. What is the capacity of a truck and of a dray?

3. A man and a boy can do in 18 days a piece of work which 5 men and 9 boys can do in 3 days. In how many days can 1 man do the work? 1 boy?

4. Two sums are put at interest at 5% and 6% respectively. The annual income from both together is $100. If the first sum had yielded 1% more and the second 1% less, the annual income would have decreased by $2. Find each sum.

5. If $ax + by = 2$ is satisfied by $x = 2$ and $y = 3$ and also by $x = 6$ and $y = 5$, what values must a and b have?

6. A bag weighing 18 ounces contains two sizes of steel balls — ounce balls and $\frac{3}{8}$-ounce balls. There are 23 balls in all. Find the number of balls of each size.

7. A man has $4.50 in dimes and quarters. If he has 36 coins in all, how many has he of each?

8. A man can walk 4 miles per hour. He reaches a point 20 miles from his starting point in three hours, having been taken part of the way by a stage traveling 12 miles per hour. How far did the stage carry him?

9. Two automobiles 25 miles apart travel toward each other and meet in 1 hour. If they had both traveled in the same direction, the faster would have overtaken the slower in 5 hours. Find the rate of each.

10. An aëroplane makes a flight in 3 hours. If the distance had been half again as great, the aëroplane would have been forced to travel 50 miles per hour faster in order to cover it in the same time. Find the distance and the speed of the aëroplane.

11. The sum of three numbers is 108. The sum of one third the first, one fourth the second, and one sixth the third is 25. Three times the first added to four times the second and six times the third is 504. Find the numbers.

12. The sum of three numbers is 217. The quotient of the first by the second is 5, which is also the quotient of the second by the third. Find the numbers.

13. If the tens' and units' digits of a three-digit number be interchanged, the resulting number is 27 less than the given number. If the same interchange is made with the tens' and hundreds' digits, the resulting number is 180 less than the given number. The sum of the digits is 14. Find the number.

14. The sum of two sides which meet at one of the vertices of a quadrilateral is 20 feet. The sum of the two which meet at the next vertex is 27 feet. The sums of the two pairs of opposite sides are 23 feet and 29 feet respectively. Find each side. (Two solutions.)

15. Two chairs cost h dollars. The first cost m cents more than the second. Find the cost of each in cents.

16. Find two numbers whose sum is a and whose difference is b.

17. Find two numbers whose sum is $a + b$ and whose difference is $a - b$.

18. Two relays of messengers carry a message k miles. The first relay travels c miles further than the second. How far does each go?

19. A man has a dollars and b cents in dimes and quarters. If he has c coins in all, how many of each kind has he?

20. A man has a dollars in quarters and nickels, with b more quarters than nickels. How many of each has he?

21. A man rows m miles downstream in t hours and returns in a hours. Find his rate in still water and the rate of the river.

22. A 10-gallon mixture of 45% alcohol is to be made from one of 95% alcohol and another of 15% alcohol. How many gallons of each will be required to make the desired mixture?

HINT. Let x and y = the number of gallons of 95% and 15% alcohol respectively. Then $\frac{.95x + .15y}{10} = \frac{45}{100}$, and $x + y = 10$.

23. A chemist has the same acid in two strengths. Eight liters of one acid solution mixed with 12 liters of another gives a mixture 84% pure, and 3 liters of the first mixed with 2 liters of the second gives a mixture 86% pure. Find the per cent of purity of each acid.

CHAPTER XXIII

EXPONENTS

157. Fundamental laws of exponents. The four laws of exponents used in the preceding chapters are

I. Law of Multiplication,

$$x^a \cdot x^b = x^{a+b}.$$

$x^2 \cdot x^3 = x^5$

If a and b are positive integers, we have

$$x^a = x \cdot x \cdot x \cdot x \cdots \text{to } a \text{ factors,}$$

and

$$x^b = x \cdot x \cdot x \cdots \text{to } b \text{ factors.}$$

Hence

$$\begin{aligned} x^a \cdot x^b &= (x \cdot x \cdot x \cdots \text{to } a \text{ factors}) \times (x \cdot x \cdot x \cdots \text{to } b \text{ factors}) \\ &= x \cdot x \cdot x \cdots \text{to } a + b \text{ factors} \\ &= x^{a+b} \text{ by the definition of an exponent.} \end{aligned}$$

II. Law of Division,

$$x^a \div x^b = x^{a-b}.$$

Again, if a and b are positive integers, we have

$$x^a \div x^b = \frac{x^a}{x^b} = \frac{x \cdot x \cdot x \cdots \text{to } a \text{ factors}}{x \cdot x \cdot x \cdots \text{to } b \text{ factors}}.$$

If b is less than a, the b factors of the denominator may be canceled with b factors of the numerator, leaving $a - b$ factors in the numerator.

Hence

$$\frac{x^a}{x^b} = x^{a-b}.$$

If b is greater than a,

$$\frac{x^a}{x^b} = \frac{1}{x^{b-a}}.$$

III. Law of Involution, or raising to a power,

$$(x^a)^b = x^{ab}.$$

$(x^2)^3 = x^2 \cdot x^2 \cdot x^2 = x^6$

As before, if a and b are positive integers, we have

$$\begin{aligned}(x^a)^b &= x^a \cdot x^a \cdot x^a \cdots \text{to } b \text{ factors}\\ &= x^{a+a+a+\cdots} \cdots \text{to } b \text{ terms of the exponent}\\ &= x^{ab}.\end{aligned}$$

IV. Law of Evolution, or extraction of roots,

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}.$$

Law I may be stated more completely thus:

$$x^a \cdot x^b \cdot x^c \cdots = x^{a+b+c+\cdots}.$$

Law III includes the more general forms:

(*a*) $$(x^a y^b)^c = x^{ac} y^{bc}.$$

(*b*) $$((x^a)^b)^c \cdots = x^{abc\cdots}.$$

When Laws I, II, and III were used in previous work in multiplication and division, we always assumed that a and b were positive integers and, in Law II, that a was greater than b. In the work on radicals (see pages 251–252) the meaning of an exponent was extended so as to include fractional exponents, as defined by Law IV. Though Laws I–IV have thus far been restricted to positive integers and fractions, they hold, nevertheless, for any irrational values of a and b. This fact will be assumed without proof. We shall now explain the meaning which, according to these laws, must be given to a zero or to a negative exponent.

158. Meaning of zero as an exponent. From Law II,

$$x^4 \div x^4 = x^{4-4} = x^0.$$

But $$x^4 \div x^4 = \frac{x^4}{x^4} = 1.$$

Therefore $$x^0 = 1.$$

More generally, $$x^a \div x^a = x^{a-a} = x^0,$$

and, as before, $$x^0 = 1.$$

That is, any number (except zero) whose exponent is zero is equal to 1. Hence $4^0 = (\frac{2}{3})^0 = (-6)^0$, for each equals 1. Again, if x is not zero, $(5x)^0 = 1$, and if x is not 1, $(x^2 - 2x + 1)^0 = 1$.

159. Meaning of a negative exponent. From Law II,

$$a^3 \div a^5 = a^{3-5} = a^{-2}.$$

$\frac{1}{a^{-2}} = \frac{1}{\frac{1}{a^2}}$

Obviously, $$a^3 \div a^5 = \frac{a^3}{a^5} = \frac{1}{a^2}.$$

Therefore a^{-2} is another way of writing $\frac{1}{a^2}$.

Then $$4^{-3} = \frac{1}{4^3} = \frac{1}{64}.$$

Also $$a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{a^2}}.$$

In general terms, $$x^{-a} = \frac{1}{x^a}.$$

Consequently $$\frac{1}{x^{-a}} = \frac{1}{\frac{1}{x^a}} = x^a.$$

Similarly, we obtain the more general results

$$bx^{-a} = \frac{b}{x^a} \quad \text{and} \quad \frac{b}{x^{-a}} = bx^a.$$

HM $c^{-2} = \frac{1}{c^2}$

Therefore, *Any factor of the numerator of a fraction may be omitted from the numerator and written as a factor of the denominator, and vice versa, if the sign of the exponent of the factor be changed.*

It follows that any expression involving negative exponents may be written as an expression involving only positive exponents. That is to say, negative exponents are not a mathematical necessity but merely a convenience. The extension of the laws of exponents which brings with it the zero exponent and the negative exponent is an illustration of what is called the Law of Permanence of Form.

It is to be understood that the part of the rule for multiplication (page 71) and of the rule for division (page 87) which determines the exponents in the product or in the quotient applies to all numbers, whether positive or negative, integral, fractional, or literal. Hence those rules need not be restated here. $a^{1/2} + a^{1/4} + 1$

$a^{1/4}$

$a^{3/4} + a^{1/2} + a^{1/4}$

$5/6$

ORAL EXERCISES

FRACTIONAL, NEGATIVE, AND ZERO EXPONENTS

Simplify:

1. $x^{\frac{2}{3}} \cdot x^{\frac{1}{3}}$. $= x$
2. $x^{\frac{1}{2}} \cdot x^{\frac{1}{3}}$. $x^{5/6}$
3. $x^{\frac{1}{3}} \cdot x^{\frac{1}{4}}$.
4. $x^{3} \div x^{\frac{1}{3}}$. 3
5. $x^{\frac{2}{3}} \cdot x^{3}$.
6. $x^{\frac{1}{4}} \cdot x$.
7. $x^{2} \cdot x^{0}$.
8. $x^{a} \cdot x$. x
9. $x^{3} \div x^{-2}$. 5
10. $x^{\frac{1}{2}} \cdot x^{-\frac{1}{3}}$. $x^{1/6}$
11. $x^{2} \div x^{-2}$. 4
12. $x^{a} \div x^{-2}$. $a+2$
13. $(5x)^{0} \cdot 5x^{0}$. 5
14. $7x^{0} \cdot x^{3}$.
15. $x^{\frac{1}{2}} \cdot x^{a}$. x
16. $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{4}}$.
17. $(a^{\frac{1}{3}} - a^{\frac{1}{5}}) \cdot a^{\frac{1}{2}}$.
18. $(a^{\frac{1}{2}} + a^{\frac{1}{4}} + 1)a^{\frac{1}{4}}$.
19. $x^{3a-1} \cdot x^{2-2a}$. x^{a+1}
20. $x^{2a-1} \cdot x^{\frac{1}{2}}$. $x^{2a-1/2}$
21. $\sqrt{x} \cdot x$. $x^{3/2}$
22. $\sqrt{x} \div x^{2}$. x
23. $\sqrt{x} \cdot \sqrt[3]{x}$. $x^{5/6}$
24. $\sqrt{x^{3}} \div \sqrt{x}$. x
25. $\sqrt{x} \cdot \sqrt[3]{x^{2}}$. $x^{7/6}$
26. $\sqrt{x^{0}} \cdot \sqrt{x^{3}}$.
27. $x^{a-1} \cdot x^{1-a}$.

$\frac{1+2a}{2}$

$a^{-1} = \frac{1}{a}$ [redacted]

28. $(x^{\frac{1}{2}})^3 \cdot x^{\frac{1}{3}}$.
29. $(x^2)^{3a} \cdot x^{-2a}$.
30. $x^3 \div \frac{1}{x^{-2}}$.
31. $x^{5a} \div \frac{1}{x^{2a}}$.
32. $x^2 \div \frac{1}{x^{-2}}$.
33. $x^{-4} \div \frac{1}{x^{-6}}$.
34. $ax^{-3} \div a^{-2}x^2$.
35. $m^2x \div m^{-2}x^2$.

Read the following with positive exponents and simplify the results:

36. m^{-3}. $\frac{1}{m^3}$
37. $2a^{-3}$. $\frac{2}{a^3}$
38. $7x^2y^{-2}$.
39. $4a^3b^{-2}c^2$.
40. $\frac{3}{a^{-2}}$.
41. $\frac{4x}{y^{-3}}$.
42. $\frac{4c^0}{xy^{-2}}$.

$\frac{5^{-1}(ab)^0}{10^{-2}b^2}$.

$\frac{4a^{-2}b^0}{y^{-5}}$.

$\frac{12x^2y^{-1}}{2yx^{-1}}$.

46. $\frac{4a^{-2}bc^{-3}}{6a^{-2}b^{-3}c^0}$.
47. $\frac{4^{-3}r^{-6}s^6}{s^{-2}r^{-2}t^3}$.
48. $2e^x \div \frac{3}{e^{-x}}$.

64x

Arrange terms so that the exponents of one letter occur in the descending order:

49. $a^2 + a^{-1} - 6a + 3a^0$.
50. $a^2 + a^{-\frac{1}{2}} - a^{\frac{1}{2}} + a + 3a^0$.
51. $a^{\frac{1}{2}} + a^{-\frac{1}{3}} + a^{-\frac{2}{3}} + a + 3$.
52. $a^2 + \frac{1}{a^3} - 2 + \frac{1}{a} + a^3$.
53. $a^3 + \frac{1}{b^3} + \frac{a}{b^2} + \frac{b^{-4}}{a^{-4}}$.
54. $a^2 - \frac{c^3}{a^2} + \frac{a}{c^2} - \frac{3c}{a} + 5$.

EXERCISES IN MULTIPLICATION

Perform the indicated multiplication and simplify:

1. $(x + x^{\frac{1}{2}} + x^{-\frac{1}{2}})2x^{\frac{3}{2}}$.
2. $(a^{\frac{1}{2}}x + ax^{\frac{1}{2}} - a^{\frac{1}{2}}x^{-\frac{1}{2}})ax^{\frac{1}{2}}$.
3. $(x^2 - 5ax + 6a^2)a^{\frac{2}{3}}x^{-\frac{2}{3}}$.
4. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}})$.
5. $(a^{-2} + 3)(a^{-2} - 5)$.
6. $(x^{2a} - 3x^a)(x^2 - 2x)$.
7. $(a^{-1} - a)^2$.
8. $(a^2 - 2a^{-2})^3$.
9. $(a^{-1} - 2a + 3a^{-2})^2$.
10. $(e^x + e^{-x})^2$.
11. $(a^{-1} + b^{-2})\left(\frac{1}{a} - \frac{1}{b^2}\right)$.

12. $(e^{2x} - 2 + e^{-2x})^2$. 14. $(e^{2x} - 3e^{-x})^4$.

13. $(x^{-\frac{1}{2}} + 2x^{\frac{1}{2}} - 3x^{\frac{3}{2}})^2$. 15. $(a^{\frac{4}{3}} - a^{\frac{2}{3}}x + 4x^2)(a^{\frac{2}{3}} + 2x)$.

16. $(x^{\frac{1}{3}} + 2y^{\frac{1}{3}})(x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + 4y^{\frac{2}{3}})$.

17. $(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}})(b^{\frac{1}{2}} + a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}})$.

18. $(x + x^{\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})(y^{-1} + x - x^{\frac{1}{2}}y^{-\frac{1}{2}})$.

19. $(\sqrt[3]{a^5} - 5\sqrt{a})(\sqrt[3]{a^5} - 5\sqrt{a})$.

HINT. This is best written $(a^{\frac{5}{3}} - 5a^{\frac{1}{2}})(a^{\frac{5}{3}} - 5a^{\frac{1}{2}})$.

20. $(5\sqrt{c^{-5}} + \sqrt[3]{d^{-3}} - a\sqrt[5]{c^{-2}})^2$.

21. $\left(\sqrt{a} + \frac{\sqrt{b}}{c}\right)\left(\sqrt{a} - \frac{\sqrt{b}}{c}\right)^2$.

EXERCISES IN DIVISION

Perform the indicated division and simplify:

1. $x^4 \div x^6$. 5. $(x^a - 2x^{2a-1} + 3x^{3a-2}) \div x^{2a-1}$.

2. $x^3 \div x^{\frac{1}{3}}$. 6. $(6a^{3+4n} - 9a^{n-2} + 12a^{2-n}) \div 3a^{n-2}$.

3. $ax^{\frac{2}{3}} \div a^{\frac{1}{2}}x^{\frac{1}{2}}$. 7. $(x + y) \div (x^{\frac{1}{3}} + y^{\frac{1}{3}})$.

4. $\frac{ax - a^2x^3}{a^2x^{\frac{1}{2}}}$. 8. $(x - 8y) \div (x^{\frac{1}{3}} - 2y^{\frac{1}{3}})$

9. $(a^{\frac{3}{2}} + b) \div (\sqrt{a} + \sqrt[3]{b})$.

10. $(a^2 + ab^{-1} + b^{-2}) \div (a - a^{\frac{1}{2}}b^{-\frac{1}{2}} + b^{-1})$.

11. $(e^{2x} + e^{-2x} + 2) \div (e^{-x} + e^x)$.

12. $\left(3e^x + \frac{1}{e^{3x}} + e^{3x} + \frac{3}{e^x}\right) \div (e^x + e^{-x})$.

13. $(a + 2b + 2a^{\frac{2}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}}) \div (a^{\frac{1}{3}} + 2b^{\frac{1}{3}})$.

14. $(m^4 - 7m^{-2} + m^{-4} + 7m^2 + 8) \div (5 - m^{-2} + m^2)$.

15. $(9x^{5n-4} - x^{3n-2} + 2x^{2n-1}) \div (2x^{n-1} + 3x^{2n-2})$.

16. $(40ab - 25a^{-\frac{9}{5}}b^{\frac{5}{4}} - 16a^{\frac{19}{5}}b^{\frac{3}{4}}) \div (-4a^{\frac{12}{5}}b^{-\frac{5}{8}} + 5a^{-\frac{2}{5}}b^{-\frac{3}{8}})$.

Divide:

$a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

$3x^{-10} + x^{6} - 4x^{-6}$ by $2x^{-2} + x^{2} + 3x^{-6}$.

. $x^{\frac{3}{5}} - y^{\frac{3}{5}}$ by $\left[\left(x^{\frac{1}{5}} - y^{\frac{1}{5}}\right) \div \left(x^{\frac{1}{10}} + y^{\frac{1}{10}}\right)\right]$. 1st

$9m + 4m^{-1} - 13$ by $3m^{\frac{1}{2}} - 5 + 2m^{-\frac{1}{2}}$.

21. $9x^{2a} + 25x^{-4a} - 19x^{-a}$ by $5x^{-2a} + 3x^{a} - 7x^{-\frac{a}{2}}$.

333 **22.** $\left(\frac{x^{\frac{3}{2}}}{8} + \frac{27y^{\frac{3}{2}}}{64}\right) \div \left[\left(\frac{3x^{\frac{1}{2}}}{2} + \frac{9y^{\frac{1}{2}}}{4}\right)\left(64x - 96x^{\frac{1}{2}}y^{\frac{1}{2}} + 144y\right)\right]$.

NOTE. To us, who use the notation of exponents every day, it seems so simple and natural a method of expressing the product of several factors that it is difficult to understand why such a long time was necessary to develop it. But here, as in many other instances, it required a great man to discover what to us seems the most obvious relation. The man who brought the notation of exponents to its modern form was John Wallis (1616–1703), an Englishman.

Though the idea of using negative and fractional exponents had occurred to writers before Wallis, it was he who showed their naturalness, and who introduced them permanently. He also was the first to use the ordinary sign ∞ to denote infinity.

336

MISCELLANEOUS EXERCISES ON EXPONENTS

Find the numerical value of the following:

1. 3^{-2}.
2. 4^{-3}.
3. $2^{-2} \cdot 3^{-4}$.
4. $7 \cdot 7^{0} \cdot 0$.
5. $(\frac{1}{2})^{-3}$.
6. $(\frac{2}{3})^{-2} \cdot 4^{0}$.
7. $\frac{2}{3^{-2}}$.
8. $\frac{12}{4^{-1}}$.
9. $5 \cdot 2^{0} - (5 \cdot 2)^{0}$.
10. $\frac{4^{-2} \cdot 3^{-2}}{6^{-2}}$.
11. $32^{-\frac{2}{5}}$.
12. $4^{-\frac{1}{2}}$.
13. $16^{-\frac{1}{4}}$.
14. $8^{-\frac{2}{3}}$.
15. $25^{1.5}$.
16. $(-64)^{-\frac{2}{3}}$.

17. $(-125)^{-\frac{2}{3}}$.

18. $\sqrt[3]{27^{-2}}$.

19. $(\sqrt[3]{-8})^2$.

20. $(\frac{1}{2})^{-4} \cdot (\frac{1}{3})^{-3} \cdot (\frac{1}{2})^0$.

21. $(\frac{1}{6})^{-2}$.

22. $(.04)^{\frac{3}{2}}$.

23. $(.027)^{-\frac{2}{3}}$.

24. $(.00032)^{\frac{2}{5}}$.

25. $\dfrac{2^{-1}}{2^{-2} - 2^{-3}}$.

HINT. $\dfrac{2^{-1}}{2^{-2} - 2^{-3}} = \dfrac{\frac{1}{2}}{\frac{1}{2^2} - \frac{1}{2^3}}$, etc.

[redacted] $\dfrac{3^{-2} - 2^{-2}}{3^{-1} - 2^{-1}}$.

[redacted] $\dfrac{3^{-3} - 2^{-3}}{3^{-1} - 2^{-1}}$.

Write with positive exponents and simplify the results:

28. $\dfrac{2}{a^{-2} - b^{-2}}$.

HINT. $\dfrac{2}{a^{-2} - b^{-2}} = \dfrac{2}{\frac{1}{a^2} - \frac{1}{b^2}}$, etc.

29. $\dfrac{3}{a^{-1} + b^{-1}}$.

30. $\dfrac{a}{a^{-2} - b^{-2}}$.

31. $\dfrac{5se^2}{s^{-2} + e^{-2}}$.

32. $\dfrac{a^{-2}}{a^{-2} + b^{-2}}$.

33. $\dfrac{a^{-3}b^{-3}}{a^{-3} + b^{-3}}$.

34. $\dfrac{a^{-2} - b^{-2}}{a^{-1} + b^{-1}}$.

35. $\dfrac{a^{-1} + b^{-1}}{a^{-3} + b^{-3}}$.

36. $\dfrac{a^{-3} - 27^{-1}}{a^{-1} - 3^{-1}}$.

Write without a denominator and simplify the results:

37. $\dfrac{2xy}{z^3}$.

38. $\dfrac{3x}{a^{-2}b^4}$.

39. $\dfrac{4sc^{-3}}{2^{-1}s^{-2}}$.

40. $\dfrac{12a^2b^3}{4xy^2}$.

41. $\dfrac{5ac^{-2}}{c(x-y)^2}$.

42. $\dfrac{1}{5a^2(c+d)^{-3}}$.

43. $\dfrac{42m^{-n}n^{2m}}{56m^{-2n}n^{-3m}}$.

44. $\dfrac{r^{-1}s^2}{r^{-2}s^3(s-r)^3}$.

Simplify:

45. $(2^3)^3$.

46. $(2^3)^{-2}$.

47. $(2^{-3})^{-2}$.

48. $\left[\left(\frac{3}{2}\right)^{-3}\right]^2$.

49. $(x^{\frac{1}{2}})^4$.

50. $(x^{\frac{1}{3}})^{\frac{1}{2}}$.

51. $(x^{-\frac{1}{2}})^{-\frac{1}{3}}$.

52. $(5a^2)^{-3}$.

53. $(c^{-2}d)^2$.

54. $(5^0 \cdot 2^6 \cdot 3^3)^{\frac{1}{3}}$.

[redacted] $(2x^5)^0 \cdot 8 \cdot 4^{\frac{1}{2}}$.

[redacted] $(a^2b)^3(a+b)$.

$(3/)^{6}$

$\dfrac{2}{6}$

57. $(a^3)^{2x}\cdot(a^2)^{3x}$. 59. $(a^3)^{x+y}\cdot(a^2)^{y-x}$. 61. $2^2\cdot 4^2=2^?$. 6

58. $(a^{x+1})^2\cdot(a^{1-x})^2$. 60. $(x^2+xy^{-2})x^{-2}$. 62. $2^n\cdot 4^n=2^?$.

[redacted] $\dfrac{2^n\cdot 8^{2n}}{4^{3n}}=2^?$.

64. $2^n\cdot 4^{n+1}\div 2^n=2^?$.

65. $\dfrac{4^{n+1}}{2^n(4^{n-1})^n}\div\dfrac{8^{n+1}}{(4^{n+1})^{n-1}}+5=?$

336-26-27
15-56
337 63-70
79-80

2 no can never be equal unless they both have exponent equal

Solve for n:

66. $3^6\cdot 9^n=81^2$.

67. $9^n\cdot 3^3=27^n$.

68. $2^{2n+2}\cdot 4^{n+2}=8^{2n}$.

69. $81\cdot 27^n=(9^n)^{\frac{n}{2}}$.

[redacted] $(25^n)^n=\dfrac{5^{7n}}{(125)^2}$.

71. $2^{6n+3}\cdot 4^{3n+6}=(8^n)^n$.

Solve for x:

72. $x^{\frac{3}{2}}=8$. 75. $x^{-\frac{2}{3}}=4$. 78. $\frac{1}{2}x^{-\frac{2}{3}}=2$.

[redacted] $(ax^{\frac{1}{2}})^{-6}=27$.

73. $x^{-\frac{1}{2}}=2$. 76. $x^{-\frac{2}{5}}=32$. [redacted] $\dfrac{\sqrt[3]{x^{\frac{7}{5}}}}{\sqrt[3]{x^{\frac{1}{5}}}}=\dfrac{\sqrt[5]{25}}{\sqrt[5]{16}}$.

74. $x^{\frac{2}{3}}=4$. 77. $x^{\frac{3}{2}}=343$.

81. $\dfrac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x-e^{-x})^2}$.

82. $x\cdot\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}}(-2x)+(a^2-x^2)^{\frac{1}{2}}$.

83. $\left(\dfrac{x^3\cdot a\cdot x^0-a\cdot 3x^2}{(x^3)^2}\right)\div\left(\dfrac{a}{x^3}\right)$.

84. $\dfrac{x^4(3x^2)-(x^3+5)4x^3}{(x^4)^2}\div\dfrac{x^3+20}{x^4}$.

85. $\dfrac{\dfrac{(x^2-1)ax^{a-1}-x^a\cdot 2x}{(x^2-1)^2}}{\dfrac{x^a}{x^2-1}}$.

CHAPTER XXIV

SQUARE ROOT AND RADICALS

(In Part Review)

160. Square root. The **square root** of any number is one of the two equal factors whose product is the number.

161. Cube root. The cube root of any number is one of the three equal factors whose product is the number.

The square roots and cube roots of monomials have been treated on pages 113–115.

162. Principal root. For a given index the **principal root** of a number is its one real root if it has but one, or its positive real root if it has two real roots of that index.

Then the principal square root of 9 is $+3$; the principal fourth root of 16 is $+2$, not -2. The square root of -4 or -9 is *not* real; such numbers have no principal square root.

The principal cube root of 8 is 2, of -27 is -3. The principal fifth root of 32 is $+2$, of -32 is -2.

Every number has more than one root of any given index except 1. The number 8, for example, has two other cube roots besides its principal cube root 2. What they are and how they are obtained will be made clear in the chapter on Imaginaries, where the consideration of the square roots of negative numbers will also be taken up.

Unless otherwise specified, only the principal odd root of a number will be considered.

ORAL EXERCISES

Find the principal square root of the following:

1. 16.
2. 25.
3. $4a^2$.
4. $36a^6$.
5. $49t^8$.
6. $\frac{1}{4}$.
7. $\frac{1}{a^2}$.
8. $\frac{9}{a^4}$.
9. $\frac{x^2}{a^2}$.
10. x^{-4}.
11. $9x^4x^{-6}$.
12. $4x^0$.

Find the principal cube root of the following:

13. 8.
14. 64.
15. $8a^3$.
16. $64x^6$.
17. -8.
18. -64.
19. $-125a^3$.
20. $-8a^9$.
21. $216a^3$.
22. $343a^6$.
23. $-512a^3$.
24. $-27x^{-6}$.

Find the principal fourth root of the following:

25. 16.
26. 81.
27. 256.
28. a^4.
29. x^{12}.
30. $16x^4$.
31. $625x^8$.
32. $16x^{-4}$.

Give the principal root and one other root for the following:

33. The fourth root of 81; of a^4; of x^{-4}.

34. The sixth root of 64; of a^6; of x^{-6}.

35. What is the sign of the principal *odd* root of a positive number? the principal odd root of a negative number?

36. What is the sign of the principal even root of a positive number?

37. How can the fourth root of a monomial be found?

38. How can the fifth root of a monomial be found?

39. Can one obtain the fifth root of a monomial by extracting the square root of its cube root? by extracting the cube root of its square root? Explain.

163. Square root of polynomials. Extracting the square root of a number is essentially an undoing of the work of multiplication. The method of doing this was indicated in the examples in section 101.

EXERCISES

Extract the square root of the following:

1. $x^4 + 4x^3 - 2x^2 - 12x + 9.$
2. $a^6 - 10a^4 - 4a^3 + 25a^2 + 20a + 4.$
3. $t^6 + 4t^4 + 9 + 4t^5 - 6t^3 - 12t^2.$
4. $4a^4 + 9a^2t^2 + t^4 + 12a^3t - 4a^2t^2 - 6at^3.$
5. $4a^8 + 12a^4 - 7 - 24a^{-4} + 16a^{-8}.$
6. $49c^{-6} - 28c^{-4} + 74c^{-2} - 20 + 25c^2.$
7. $9x^4 - 6x^{\frac{7}{2}} + x^3 - 66x^{\frac{5}{2}} + 22x^2 + 121x.$
8. $25x^3 - 10x^2 + 90x^{\frac{7}{4}} + x - 18x^{\frac{3}{4}} + 81x^{\frac{1}{2}}.$
9. $x^4 + 6x^3 + \dfrac{29x^2}{3} + 2x + \dfrac{1}{9}.$
10. $\dfrac{4x^2}{y^2} + 7 + \dfrac{y^2}{4x^2} + \dfrac{12x}{y} - \dfrac{3y}{x}.$

Solution. Arranging terms in descending powers of x and applying the rule on page 241, we obtain the following:

$$
\begin{array}{lll}
 & \dfrac{4x^2}{y^2} + \dfrac{12x}{y} + 7 - \dfrac{3y}{x} + \dfrac{y^2}{4x^2} & \bigg| \ \dfrac{2x}{y} + 3 - \dfrac{y}{2x} \\
\left(\dfrac{2x}{y}\right)^2 = & \dfrac{4x^2}{y^2} & \\
\hline
2\left(\dfrac{2x}{y}\right) = \dfrac{4x}{y} & \quad \dfrac{12x}{y} + 7 & \\
\qquad \dfrac{4x}{y} + 3 & \quad \dfrac{12x}{y} + 9 & = \left(\dfrac{4x}{y} + 3\right)3 \\
\hline
2\left(\dfrac{2x}{y} + 3\right) = \dfrac{4x}{y} + 6 & \qquad -2 - \dfrac{3y}{x} + \dfrac{y^2}{4x^2} & \\
\qquad \dfrac{4x}{y} + 6 - \dfrac{y}{2x} & \qquad -2 - \dfrac{3y}{x} + \dfrac{y^2}{4x^2} & = \left(\dfrac{4x}{y} + 6 - \dfrac{y}{2x}\right)\left(-\dfrac{y}{2x}\right) \\
\hline
\end{array}
$$

Therefore the square roots are $\pm\left(\dfrac{2x}{y} + 3 - \dfrac{y}{2x}\right).$

11. $4a^4 + \dfrac{4a^3}{3} - \dfrac{35a^2}{9} - \dfrac{2a}{3} + 1.$

12. $\frac{25x^4}{4} - \frac{127x^2}{18} - 2x + \frac{10x^3}{3} + 2\frac{1}{4}$.

13. $\frac{9}{4}a^4 - 2a^3 + 7\frac{4}{9}a^2 - \frac{28}{9}a + \frac{49}{9}$.

14. $\frac{a^2}{9} - 18 + \frac{9}{a^2} + 9a^2 - 2 + 2a^2$.

15. $\frac{x^2}{a^2} - \frac{5a}{x} + \frac{5x}{a} + \frac{a^2}{x^2} + 4\frac{1}{4}$.

16. $\frac{a^2}{b^2} + \frac{b^2}{a^2} + a^2b^2 + 2a^2 + 2 + 2b^2$.

17. $\frac{a^4}{25c^4} + \frac{2a^2}{c^3} + \frac{117}{5c^2} - \frac{40}{a^2c} + \frac{16}{a^4}$.

18. $\frac{a^2}{4c^4} + 9 + \frac{4c^2}{25a^4} - \frac{3a}{c^2} + \frac{2}{5ac} - \frac{12c}{5a^2}$.

19. $\frac{1}{4a^4} - \frac{3x}{a^3} + \frac{9x^2}{a^2} + \frac{a^4}{25x^2} - \frac{6a}{5} + \frac{1}{5x}$.

Find the first four terms in the square root of the following:

20. $1 + 2x$.

21. $\frac{25}{9} + a^2$.

164. Square root of arithmetical numbers. For explanations, examples, and rule see pages 243–246.

EXERCISES

Find the positive square root of the following:

1. 6241.

2. 9216.

3. 15,129.

4. 56,169.

5. 1.4641.

6. 216.09.

7. 988,036.

8. 7,049,025.

9. 3.9601.

Find to three decimal places the positive square root of the following:

10. 7.

11. .63.

12. .01235.

13. .96384.

14. $\frac{5}{7}$.

15. $4\frac{2}{9}$.

16. $\frac{10}{7}$.

17. $\frac{4}{5}$.

18. $23\frac{6}{11}$.

19. $89\frac{1}{3}$.

20. Find the hypotenuse of a right triangle whose legs are 136 and 273 respectively.

21. A baseball diamond is a square 90 feet on each side. Find the distance from the home plate to second base, correct to .01 of a foot.

22. The hypotenuse of a right triangle is 207 feet, and one leg is 83 feet. Find the other leg, correct to .01 of a foot.

23. The hypotenuse and one leg of a right triangle are respectively 5471 and 4059. Find the other leg.

24. The side of an equilateral triangle is 17 inches. Find its altitude, correct to .1 of an inch.

25. Find the side of an equilateral triangle whose altitude is 15 inches, correct to .001 of an inch.

Fact from Geometry. If a, b, and c represent the sides of a triangle and s equals one half of $a + b + c$, the area of the triangle equals $\sqrt{s(s-a)(s-b)(s-c)}$.

26. Find the area of a triangle whose sides are 12, 27, and 35 inches respectively, correct to .001 of a square foot.

27. By the method of Exercise 26 find to .01 of a square inch the area of a triangle each side of which is 22 inches.

28. Find to two decimals the radius of a circle whose area is 70 square feet.

29. Find to two decimals the diagonal of a room whose dimensions in feet are 15, 22, and 28.

30. Find to two decimals the diagonal of a cube whose edge is 8 feet.

31. A room is 24 feet by 40 feet by 14 feet. What is the length of the shortest broken line from one lower corner to the diagonally opposite upper corner, the line to be on the walls or the floor, but not through the air?

165. Irrational numbers. Any real number which is not rational is **irrational.**

If a number under a radical sign is such that the root indicated cannot be exactly obtained, the radical represents an irrational number.

For example, $\sqrt{7}$ and $\sqrt[3]{4}$ are irrational. Approximate values for these are given on page 502.

A repeating decimal, though endless, is not an irrational number, for any repeating decimal can be expressed as a common fraction, and is therefore rational.

Thus the repeating decimal .272727 ⋯ is not irrational, as it exactly equals $\frac{3}{11}$. Similarly, .2857142857142 ⋯ exactly equals $\frac{2}{7}$, etc.

NOTE. The number $\frac{2}{7}$ reduced to a decimal repeats the digits in groups of six each, and the mere fact that a decimal does so repeat is proof that it is a rational number. On the other hand, the number π is known to be irrational, and its value has been computed to 707 decimals, showing, of course, no repetition. The fact that it does not repeat in 700 digits is, however, no proof that π is irrational, for decimals with even more than that many digits do repeat. For example, the fraction $\frac{10000}{7699}$ equals the decimal 1.29 +, which repeats in groups of 7698 digits each.

166. Imaginary numbers. An indicated *square* root of a *negative* number is called an **imaginary** number.

Thus $\sqrt{-4}$, $\sqrt{-7}$, and $\sqrt{-12}$ are imaginary numbers. And $3+\sqrt{-1}$ is also imaginary, though, as will be seen later (Chapter XXVIII), such numbers are better called complex numbers.

167. Classification of numbers. All the numbers of algebra then may be placed in one or the other of two classes: *real* numbers and *imaginary* numbers.

Real numbers, as we have seen, are of two kinds, *rational* numbers and *irrational* numbers. (See page 250.)

168. Surd. A **surd** is an irrational number in which the radicand is rational.

Thus $\sqrt{3}$, $\sqrt[3]{9}$, etc. are surds. But $\sqrt{2+\sqrt{3}}$ and $\sqrt{\pi}$ are not surds.

169. The algebraic sign of a radical. The square root of 4 is both $+2$ and -2. The symbol $\sqrt{4}$, however, signifies only $+2$, the principal root (section 162). Similarly, $\sqrt[4]{81}$ is $+3$ and $\sqrt[6]{64}$ is $+2$. But $-\sqrt{9}$ is -3, and $-\sqrt[4]{16}$ is -2. The symbol $\pm\sqrt{25}$ denotes both $+5$ and -5. Further, $+\sqrt[3]{27}=+3$, $-\sqrt[3]{27}=-3$, and $-\sqrt[3]{-27}=+3$.

The foregoing remarks apply also to fractional exponents. Thus $4^{\frac{1}{2}}=+2$ only, and $81^{\frac{1}{4}}=+3$ only, etc. It should be noted that these statements really define the meaning of such symbols as $\sqrt{\ }$, $\sqrt[4]{\ }$, $\sqrt[6]{\ }$, etc. Such an understanding as this avoids all the ambiguity which would arise if $\sqrt{16}$ meant both $+4$ and -4. The distinctions here made are especially needed in radical equations.

Moreover, without the limitation stated the following relation is not always true:

$$\sqrt[b]{x^a}=(\sqrt[b]{x})^a.$$

For example, we would have

$$(9^{\frac{1}{2}})^6=(\pm 3)^6=729,$$

while

$$(9^6)^{\frac{1}{2}}=(590041)^{\frac{1}{2}}=\pm 729.$$

ORAL EXERCISES

Find the numerical value of the following:

1. $\sqrt{4}$.
2. $-\sqrt{9}$.
3. $\sqrt[3]{8}$.
4. $\sqrt[3]{-8}$.
5. $-\sqrt[4]{16}$.
6. $\sqrt[3]{-125}$.
7. $-\sqrt[5]{-32}$.
8. $\sqrt[6]{64}$.
9. $-\sqrt[4]{625}$.
10. $8^{\frac{1}{3}}$.
11. $49^{\frac{1}{2}}$.
12. $8^{\frac{2}{3}}$.
13. $27^{\frac{2}{3}}$.
14. $16^{\frac{3}{4}}$.
15. $(\frac{4}{9})^{\frac{1}{2}}$.
16. $(-64)^{\frac{1}{3}}$.

Read in radical form:

17. $x^{\frac{3}{2}}$.

18. $(at)^{\frac{3}{2}}$.

19. $(3t)^{\frac{1}{3}}$.

20. $3t^{\frac{1}{2}}$.

21. $3rx^{\frac{1}{2}}$.

22. $2a^{\frac{1}{2}}x^{\frac{1}{3}}$.

23. $5ax^{\frac{3}{4}}$.

24. $x^{\frac{1}{n}}$.

25. $2^{\frac{1}{3}}x^{\frac{1}{a}}$.

Read with fractional exponents:

26. $\sqrt{a^5}$.

27. $\sqrt{ax^3}$.

28. $\sqrt[3]{a^4}$.

29. $\sqrt[3]{a^2x^4}$.

30. $2\sqrt{a}$.

31. $\sqrt[3]{2a}$.

32. $\sqrt[3]{(a+x)^2}$.

33. $\sqrt{(3x-a)^4}$.

34. $\sqrt[3]{x^2(y-1)}$.

35. What are the two square roots of 36?

36. What are two fourth roots of 16? of 81? of 625?

37. What is the value of $\sqrt[4]{16}$? of $\sqrt[4]{81}$? of $\sqrt[4]{625}$?

38. What are two sixth roots of 64?

39. Distinguish between a rational and an irrational number.

40. Which of the numbers 8, $\frac{2}{3}$, 343, $\sqrt{4}$, $\sqrt{3}$, and π ($\pi = 3.14159+$) are rational? Which are irrational?

41. Give a geometrical illustration of an irrational number by means of a right triangle.

42. Is a radical always a surd? Illustrate.

43. Is a surd always a radical? Illustrate.

44. Distinguish between a surd and a radical.

45. Which of the numbers $\sqrt{3}$, $\sqrt{4}$, $\sqrt[3]{27}$, $\sqrt{\sqrt[3]{6}}$, $\sqrt{2+\sqrt{3}}$, and $\sqrt{2\pi}$ are surds? Which are radicals?

46. What is the principal root of $\sqrt{4}$, $\sqrt[3]{8}$, and $\sqrt[3]{-8}$?

47. Name the order of $\sqrt{6}$; of $a^{\frac{1}{3}}$; of $\sqrt[3]{5}$; of $c^{\frac{2}{3}}$; of $\sqrt[4]{m^3}$.

48. Give an example of (*a*) a real number; (*b*) an imaginary number; (*c*) a rational number; (*d*) an irrational number; (*e*) a radical; (*f*) a surd; (*g*) an index; (*h*) a radicand; (*i*) the principal odd root of a positive number; (*j*) the principal even root of a positive number; (*k*) the principal odd root of a negative number.

170. Simplification of radicals. The rules of pages 254 and 256 and their applications to the work of section 109 may be summarized and illustrated as follows:

A radical is in its simplest form when the radicand

I. *Is integral.*

II. *Contains no rational factor raised to a power which is equal to, or greater than, the order of the radical.*

III. *Is not raised to a power, unless the exponent of the power and the index of the root are prime to each other.*

For the meaning of I, II, and III study carefully the following examples. Particular attention is called to III and the examples under it, as this type was not treated in Chapter XX.

EXAMPLES

Examples of I:

1. $\sqrt{\frac{3}{2}}=\sqrt{\frac{6}{4}}=\sqrt{\frac{1}{4}\cdot 6}=\sqrt{\frac{1}{4}}\sqrt{6}=\frac{1}{2}\sqrt{6}.$

2. $6\sqrt{\frac{1}{3}}=6\sqrt{\frac{3}{9}}=6\sqrt{\frac{1}{9}\cdot 3}=6\cdot\frac{1}{3}\sqrt{3}=2\sqrt{3}.$

3. $\sqrt{\frac{3}{5x}}=\sqrt{\frac{15x}{25x^2}}=\sqrt{\frac{1}{25x^2}\cdot 15x}=\frac{1}{5x}\sqrt{15x}.$

Examples of II:

1. $\sqrt{4x^5}=\sqrt{4x^4\cdot x}=\sqrt{(2x^2)^2\cdot x}=2x^2\sqrt{x}.$

2. $5\sqrt[3]{24x^5}=5\sqrt[3]{8x^3\cdot 3x^2}=5\sqrt[3]{(2x)^3\cdot 3x^2}=10x\sqrt[3]{3x^2}.$

3. $\sqrt{16-8\sqrt{2}}=\sqrt{4(4-2\sqrt{2})}=2\sqrt{4-2\sqrt{2}}.$

Examples of III:

1. $\sqrt[4]{4}=\sqrt[4]{2^2}=2^{\frac{2}{4}}=2^{\frac{1}{2}}=\sqrt{2}.$

2. $\sqrt[6]{9}=\sqrt[6]{3^2}=3^{\frac{2}{6}}=3^{\frac{1}{3}}=\sqrt[3]{3}.$

3. $\sqrt[4]{a^2b^4}=a^{\frac{2}{4}}b^{\frac{4}{4}}=a^{\frac{1}{2}}b=b\sqrt{a}.$

A radical of the second order is simplified by the use of the

Rule. *Separate the radicand into two factors one of which is the greatest perfect square which it contains. Then take the square root of this factor and write it as the coefficient of a radical which has the other factor as radicand.*

If the original radical has a coefficient other than the number 1, multiply the result obtained above by this coefficient.

A similar rule holds for simplifying radicals involving the cube root and roots of higher orders.

EXERCISES

Simplify:

1. $\sqrt{18}$.
2. $\sqrt{28}$.
3. $\sqrt{50}$.
4. $\sqrt{52}$.
5. $\sqrt{68}$.
6. $\sqrt{108}$.
7. $\sqrt{192}$.
8. $2\sqrt{45}$.
9. $\sqrt[3]{16}$.
10. $4\sqrt[3]{54}$.
11. $\sqrt[4]{48}$.
12. $3\sqrt[5]{64}$.
13. $\sqrt{a^3}$.
14. $\sqrt{ax^3}$.
15. $\sqrt[3]{a^2x^5}$.
16. $5\sqrt[3]{16a^4}$.

17. $\sqrt{\frac{3}{10}}$.

Solution. $\sqrt{\frac{3}{10}} = \sqrt{\frac{30}{100}} = \sqrt{\frac{1}{100} \cdot 30} = \frac{1}{10}\sqrt{30}$.

18. $\sqrt{\frac{2}{5}}$.
19. $\sqrt{\frac{3}{7}}$.
20. $\sqrt[3]{\frac{1}{4}}$.
21. $\sqrt[3]{\frac{2}{9}}$.

22. $\sqrt{\frac{1}{a}}$.

Solution. $\sqrt{\frac{1}{a}} = \sqrt{\frac{a}{a^2}} = \sqrt{\frac{1}{a^2} \cdot a} = \frac{1}{a}\sqrt{a}$.

23. $\sqrt{\frac{1}{2x}}$.
24. $\sqrt{\frac{1}{a^3}}$.
25. $\sqrt[3]{\frac{1}{a^2}}$.
26. $\sqrt[3]{\frac{1}{ax^3}}$.
27. $\sqrt[3]{-\frac{3}{4}}$.
28. $\sqrt{1-(\frac{1}{3})^2}$.
29. $\sqrt{3^2-(\frac{3}{2})^2}$.
30. $\sqrt{s^2-\left(\frac{s}{2}\right)^2}$.
31. $\sqrt{s^3+\left(\frac{s}{2}\right)^3}$.
32. $\sqrt{\left(\frac{a+1}{2}\right)^2-a}$.

33. $\sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2-1}.$

34. $\sqrt{4-8\sqrt{3}}.$

HINT. $\sqrt{4-8\sqrt{3}}=$
$\sqrt{4(1-2\sqrt{3})}.$

35. $\sqrt{36+18\sqrt{5}}.$

36. $\sqrt[3]{16-8\sqrt{3}}.$

37. $\sqrt[3]{54-9\sqrt{18}}.$

38. $\sqrt{R^2+3R^2\sqrt{5}}.$

39. $\sqrt{R^2-\frac{R^2}{2}\sqrt{3}}.$

40. $\sqrt[4]{4}.$

Solution. $\sqrt[4]{4}=\sqrt[4]{2^2}=2^{\frac{2}{4}}$
$=2^{\frac{1}{2}}=\sqrt{2}.$

41. $\sqrt[4]{9}.$

42. $\sqrt[4]{a^2b^4}.$

43. $\sqrt[4]{64a^2b^4}.$

44. $\sqrt[6]{9a^2}.$

45. $\sqrt[6]{\frac{4a^6}{25}}.$

Express entirely under the radical sign:

46. $2\sqrt{7}.$

Solution. $2\sqrt{7}=$
$\sqrt{4}\sqrt{7}=\sqrt{28}.$

47. $3\sqrt{5}.$

48. $2\sqrt[3]{8}.$

49. $a\sqrt{a}.$

50. $2c\sqrt[3]{c^2}.$

51. $4\sqrt[3]{\frac{1}{4}}.$

52. $\frac{a}{3}\sqrt[3]{\frac{9}{a^2}}.$

53. $e^x\sqrt{e^x+e^{-x}}.$

54. $(a+1)\sqrt{\frac{1}{a^2-1}}.$

55. $\frac{x-3a}{5}\sqrt[3]{\frac{125}{(x-3a)^2}}.$

Express in simplest form with one radical sign:

56. $\sqrt{\sqrt{2}}.$

Solution. $\sqrt{\sqrt{2}}=\sqrt{2^{\frac{1}{2}}}$
$=2^{\frac{1}{4}}=\sqrt[4]{2}.$

57. $\sqrt{\sqrt{a}}.$

58. $\sqrt[3]{\sqrt{a}}.$

59. $\sqrt{\sqrt[3]{a^2}}.$

60. $\sqrt{\sqrt{x^8}}.$

61. $\sqrt[3]{\sqrt{x^6}}.$

62. $\sqrt[3]{\sqrt{8a^2x}}.$

63. $\sqrt{3\sqrt{3}}.$

64. $\sqrt[4]{\sqrt{8}}.$

65. $2\sqrt[3]{2\sqrt[3]{2}}.$

66. $\sqrt[a]{\sqrt[n]{x^c}}.$

67. $\sqrt[\frac{m}{n}]{x^{\frac{a}{n}}}$

Find by the formula of Exercise 26, page 342, the areas of the triangles whose sides are

68. 6, 8, and 10.

69. 7, 24, and 25.

70. 33, 56, and 65.

71. 104, 153, and 185.

171. Addition and subtraction of radicals. **Similar radicals** are radicals of the same order, with radicands which are identical or which can be made so by simplification.

The sum or the difference of similar radicals can be expressed as one term, while the sum or difference of dissimilar radicals can only be indicated.

EXERCISES

Simplify and collect:

1. $\sqrt{8}+\sqrt{18}$.

Solution. $\sqrt{8}+\sqrt{18}=$
$2\sqrt{2}+3\sqrt{2}=5\sqrt{2}$.

2. $\sqrt{20}+3\sqrt{5}$.

3. $\sqrt{50}+\sqrt{98}-\sqrt{32}$.

4. $\sqrt{12}+5\sqrt{75}-2\sqrt{27}$.

5. $\sqrt{75}+3\sqrt{147}-\sqrt{12}$.

6. $2\sqrt{54}+\sqrt{24}-\sqrt{96}$.

7. $\sqrt[3]{16}+\sqrt[3]{54}-3\sqrt[3]{2}$.

8. $\sqrt[3]{192}-4\sqrt[3]{24}+\sqrt[3]{375}$.

9. $\sqrt[3]{625}+\sqrt[3]{40}+\sqrt[3]{135}$.

10. $10\sqrt{\frac{6}{5}}-\sqrt{\frac{3}{10}}+\sqrt{\frac{15}{2}}$.

11. $a\sqrt{x^3}-\sqrt{a^2x}-5\sqrt{a^2x}$.

12. $\sqrt{x^3}+\sqrt[4]{x^2}-12\sqrt[6]{x^3}$.

13. $\sqrt{\frac{a}{x^3}}-\sqrt{\frac{a}{x^5}}+\sqrt{\frac{5x^3}{a}}$.

14. $\sqrt{\frac{6}{7}}+\sqrt{\frac{3}{14}}-\sqrt{\frac{21}{2}}$.

15. $\sqrt[4]{32x^5}+\sqrt[4]{1250x}-\sqrt[4]{512x}-\sqrt[4]{2x}$.

16. $\sqrt{(a+c)^3}-c\sqrt[4]{(a+c)^2}+2c\sqrt[6]{(a+c)^3}$.

17. $\sqrt[3]{(a-c)^4}+c\sqrt[6]{a^2-2ac+c^2}+(a+c)\sqrt[3]{a-c}$.

18. $\sqrt{\frac{a}{c}}-\sqrt{\frac{c}{a}}+\sqrt{\frac{a^2+c^2}{ac}+2}-\sqrt{\frac{a^2+c^2}{ac}-2}$.

19. $\sqrt[3]{24}+\sqrt[3]{(3a+9)(a+3)^2}-\sqrt[3]{81}+a\sqrt[6]{9}-4\sqrt[3]{3}$.

20. $2\sqrt{9a^3-9a^2b}-3\sqrt{9ab^2-9b^3}+\sqrt{(a^2-b^2)(a+b)}$.

21. $(a-b)\sqrt{\frac{a+b}{a-b}}+\sqrt{25a^2-25b^2}+\frac{a+b}{a-b}\sqrt{\frac{36ab^2-36b^3}{a+b}}$.

172. Multiplication of real radicals. Real radicals of the same order are multiplied as follows:

Example 1. Multiply $2\sqrt{x} - 3\sqrt{a} - 4\sqrt{ax}$ by $2\sqrt{ax}$.

Solution.

$$\begin{array}{l} 2\sqrt{x} - 3\sqrt{a} - 4\sqrt{ax} \\ \underline{2\sqrt{ax} \qquad\qquad\qquad\quad} \\ 4x\sqrt{a} - 6a\sqrt{x} - 8ax \end{array}$$

Real radicals of different order are multiplied as follows:

Example 2. Multiply $\sqrt{n}$ by $\sqrt[3]{x}$.

Solution.

$$\sqrt{n} = n^{\frac{1}{2}} = n^{\frac{3}{6}} = \sqrt[6]{n^3}.$$

$$\sqrt[3]{x} = x^{\frac{1}{3}} = x^{\frac{2}{6}} = \sqrt[6]{x^2}.$$

Then $\sqrt{n}\cdot\sqrt[3]{x} = \sqrt[6]{n^3}\cdot\sqrt[6]{x^2} = \sqrt[6]{n^3x^2}.$

The method of multiplying real radicals is stated in the

Rule. *If necessary, reduce the radicals to the same order.*

Find the products of the coefficients of the radicals for the coefficient of the radical part of the result.

Multiply together the radicands and write the product under the common radical sign.

Reduce the result to its simplest form.

The preceding rule does not hold for the multiplication of imaginary numbers. This case is discussed in Chapter XXVIII.

EXERCISES

Perform the indicated multiplication and simplify the products:

1. $\sqrt{12}\sqrt{18}$.
2. $\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{3}{4}}$.
3. $\sqrt{\frac{4}{5}}\cdot\sqrt{\frac{5}{6}}\cdot\sqrt{\frac{3}{4}}$.
4. $(\sqrt{2} - 3\sqrt{5})\sqrt{5}$.
5. $(\sqrt{3} - 2\sqrt{2})(\sqrt{2} - \sqrt{3})$.
6. $(\sqrt{5} - 3\sqrt{2})(2\sqrt{5} - \sqrt{3})$.
7. $(\sqrt{a} - \sqrt{ax})(\sqrt{a} + 2\sqrt{ax})$.
8. $(3\sqrt{5} - \sqrt{2})(3\sqrt{5} + \sqrt{2})$.
9. $(4\sqrt{5} + 2\sqrt{7})(4\sqrt{5} - 2\sqrt{7})$.

[illegible] $(\sqrt{3a} - \sqrt{2a})^2$. **11.** $(2\sqrt{3x-1})^2$. [illegible] $(\sqrt{x} - \sqrt{x-2})^2$.

13. $(\sqrt{5} - \sqrt{3} - \sqrt{2})(\sqrt{5} + \sqrt{3} - \sqrt{2})$.

14. $(3\sqrt{2} + 2\sqrt{3} + \sqrt{30})(2\sqrt{2} + 2\sqrt{3} - 2\sqrt{5})$.

15. $\left(R - \frac{R}{2}\sqrt{3}\right)\left(2R + \frac{3R}{2}\sqrt{3}\right)$.

16. $\left(\frac{R}{2}\right)^2 + \left(\frac{R}{2} - \frac{R}{2}\sqrt{3}\right)^2$. [illegible] $R^2 - \left(\frac{R}{2}\sqrt{5} - \frac{R}{2}\right)^2$.

Square:

18. $\sqrt{2} - \sqrt{x-3}$. **20.** $2\sqrt{x} - 3\sqrt{2x+1}$.

19. $\sqrt{x-3} + \sqrt{x+5}$. **21.** $3\sqrt{x-1} + 2\sqrt{2-x}$.

Perform the indicated multiplication:

22. $(a + \sqrt{a+b})(a - \sqrt{a+b})$. $= a^2 - (a$ [illegible]

23. $(\sqrt{2x-3} - \sqrt{3x})(\sqrt{2x-3} + \sqrt{3x})$.

Express as radicals of same order:

24. $\sqrt{2}$ and $\sqrt[3]{3}$. **25.** $\sqrt[4]{ax}$ and $\sqrt{a}$. **26.** $\sqrt[3]{9}$ and $\sqrt{8}$.

Multiply the following:

27. $\sqrt{3}, \sqrt[3]{3}$. **30.** $\sqrt[3]{12}, \sqrt{\frac{1}{8}}$. **32.** $\sqrt{\frac{x}{a}}, \sqrt[3]{\frac{a}{x}}$.

28. $\sqrt{a}, \sqrt[3]{a}$.

29. $\sqrt[4]{a^3}, \sqrt{a}$. **31.** $\sqrt[3]{a^2}, \sqrt{a^3}$. **33.** $\sqrt[4]{2x^3}, \sqrt{3x}$.

173. Division of radicals. Division of radicals coefficient by coefficient and radicand by radicand is often possible.

Thus $6\sqrt{5} \div 3\sqrt{3} = 2\sqrt{\frac{5}{3}} = \frac{2}{3}\sqrt{15}$,

and $3\sqrt{ax} \div 2\sqrt{x} = \frac{3}{2}\sqrt{a}$.

Direct division of radicals when the divisor is a radical expression with more than one term is usually very difficult. In such cases a rationalizing factor of the denominator is used. We then carry out the operation of division indirectly by resorting to multiplication.

174. Rationalizing factor. Division of radicals by the use of a rationalizing factor is treated on pages 266–268. Some additional examples of its use are given here.

Rationalizing factors are used in division of radicals as follows:

Example 1. $\sqrt{6} \div \sqrt{5} = \dfrac{\sqrt{6}\sqrt{5}}{\sqrt{5}\sqrt{5}} = \dfrac{\sqrt{30}}{5}.$

Example 2.
$$(6\sqrt{2} - 15\sqrt{3}) \div 3\sqrt{5} = \frac{(6\sqrt{2} - 15\sqrt{3}) \cdot \sqrt{5}}{3\sqrt{5} \cdot \sqrt{5}} = \frac{2\sqrt{10} - 5\sqrt{15}}{5}.$$

Example 3.
$$\begin{aligned}(\sqrt{3} + \sqrt{2}) \div (\sqrt{5} - \sqrt{3}) &= \frac{(\sqrt{3} + \sqrt{2})(\sqrt{5} + \sqrt{3})}{(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})} \\ &= \frac{\sqrt{15} + \sqrt{10} + 3 + \sqrt{6}}{5 - 3} \\ &= \tfrac{1}{2}(\sqrt{15} + \sqrt{10} + 3 + \sqrt{6}).\end{aligned}$$

It should be noted that the rule on page 267 applies in all cases, while the rule for direct division fails when dividing a real radical by an imaginary number.

EXERCISES

Find a simple rationalizing factor for:

1. $\sqrt{7}$.
2. $3\sqrt{5}$.
3. $\sqrt{8}$.
4. $\sqrt[3]{4}$.
5. $\sqrt[3]{5}$.
6. $\sqrt[4]{8}$.
7. $\sqrt{3} - 2$.
8. $3\sqrt{6} - 2\sqrt{11}$.
9. $\sqrt{3a} - \sqrt{2x}$.
10. $\sqrt{x-3} - 2\sqrt{3x}$.
11. $\sqrt{a-b} + \sqrt{a+b}$.
12. $\sqrt{8} + \sqrt{2} - \sqrt{5}$.

Perform the indicated division:

13. $\sqrt{8} \div \sqrt{2}$.
14. $6\sqrt{10} \div \sqrt{5}$.
15. $\sqrt{8} \div \sqrt{24}$.
16. $\sqrt{2ax} \div \sqrt{3a^2x}$.
17. $(2\sqrt{10} - 3\sqrt{5}) \div 2\sqrt{5}$.
18. $(\sqrt{ax^2} - \sqrt{a^2x}) \div \sqrt{ax}$.
19. $8 \div 4\sqrt{3}$.
20. $24 \div 3\sqrt{3}$.

21. $\sqrt{3} \div \sqrt[3]{2}$.

Solution. $\dfrac{\sqrt{3}}{\sqrt[3]{2}} = \dfrac{\sqrt{3}\cdot\sqrt[3]{4}}{\sqrt[3]{2}\cdot\sqrt[3]{4}} = \dfrac{\sqrt{3}\cdot\sqrt[3]{4}}{2} = \dfrac{\sqrt[6]{27\cdot 16}}{2}$.

22. $\sqrt{5} \div \sqrt[3]{3}$.

23. $\sqrt{a} \div \sqrt[4]{2}$.

24. $\sqrt[4]{8} \div \sqrt{2}$.

25. $\sqrt[3]{a} \div \sqrt[6]{a}$.

26. $\sqrt[3]{\frac{1}{4}} \div \sqrt{\frac{1}{2}}$.

27. $\sqrt{3} \div (\sqrt{3} - 2)$.

28. $\sqrt{5} \div (\sqrt{5} + \sqrt{2})$.

29. $(2\sqrt{3} + \sqrt{5}) \div (\sqrt{3} - \sqrt{5})$.

Change to respectively equivalent fractions having rational denominators:

30. $\dfrac{\sqrt{3}}{2\sqrt{6} - \sqrt{3}}$.

32. $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}}$.

34. $\dfrac{\sqrt{a+x}}{\sqrt{5} - \sqrt{a+x}}$. Simplify

31. $\dfrac{2\sqrt{5} + 3\sqrt{7}}{3\sqrt{5} - 2\sqrt{7}}$.

33. $\dfrac{\sqrt{x-3} + \sqrt{3}}{\sqrt{x-3} - \sqrt{3}}$.

35. $\dfrac{4}{\sqrt{2 - \sqrt{2}}}$.

Perform the indicated division:

36. $(\sqrt{10} - \sqrt{5}) \div (\sqrt{10} + \sqrt{5})$. Find [illegible]

37. $(2 - \sqrt{3} + \sqrt{2}) \div (\sqrt{3} + \sqrt{2})$.

38. $(\sqrt{5} + \sqrt{7} - \sqrt{2}) \div (\sqrt{5} - \sqrt{7})$. Simplify

39. Is there any distinction between the meaning of the direction before Exercise 30 and of that before Exercise 36?

40. Does $3 - \sqrt{7}$ satisfy $x^2 - 6x + 2 = 0$?

41. Does $\dfrac{7 - \sqrt{3}}{2}$ satisfy $2x^2 - 75x + 161 = 0$?

42. Does $\frac{1}{6}(5 \pm \sqrt{109})$ satisfy $3x^2 - 5x - 7 = 0$?

175. Square root of surd expressions. The square of a binomial is usually a trinomial. However, the result of squaring a binomial of the form $\sqrt{a} + \sqrt{b}$ is a binomial if a and b are rational numbers. Thus

$$(\sqrt{7} - \sqrt{3})^2 = 7 - 2\sqrt{21} + 3 = 10 - 2\sqrt{21}.$$

In $10 - 2\sqrt{21}$, 10 is the sum of 7 and 3, and 21 is the product of 7 and 3. These relations and the fact that the coefficient of the radical $\sqrt{21}$ is 2 enable us to find the square root of many expressions of the form $a \pm \sqrt{b}$ by writing each in the form of $x \pm 2\sqrt{xy} + y$ and then extracting the square root of the trinomial square as follows:

Example. Extract the square root of $9 - \sqrt{56}$.

Solution. $9 - \sqrt{56} = 9 - 2\sqrt{14}$.

Then $9 - 2\sqrt{14} = 7 - 2\sqrt{14} + 2 = (\sqrt{7} - \sqrt{2})^2$.

Hence the square roots of $9 - \sqrt{56}$ are $\pm (\sqrt{7} - \sqrt{2})$.

EXERCISES

Find the positive square roots in Exercises 1–13:

1. $8 - 2\sqrt{15}$.
2. $5 - 2\sqrt{6}$.
3. $13 + \sqrt{48}$.
4. $7 - \sqrt{40}$.
5. $16 - 6\sqrt{7}$.
6. $17 + 12\sqrt{2}$.
7. $11 - 3\sqrt{8}$.
8. $11 - \sqrt{120}$.
9. $65x - 20\sqrt{3x^2}$.
10. $126a - 10a\sqrt{5}$.
11. $\frac{13a}{4} - \sqrt{3a^2}$.
12. $2x + 2\sqrt{x^2 - 49}$.
13. $a + \sqrt{a^2 - 1}$.
14. $\sqrt{9 + 3\sqrt{8}} = \sqrt{?} + \sqrt{?}$.
15. $\sqrt{15 - 5\sqrt{8}} = ?$
16. $\sqrt{a + \sqrt{a^2 - 4b^2}} = ?$
17. $\sqrt{m^2 + m + 2n + 2m\sqrt{m + 2n}} = ?$

Note. In the writings of one of the later Hindu mathematicians (about A.D. 1150) we find a method of extracting the square root of surds which is practically the same as that given in the text. In fact, the formula for the operation is given, apart from the modern symbols, as follows: $\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{ab}}$. The study of expressions of the type $\sqrt{\sqrt{a} \pm \sqrt{b}}$ had been carried to a most remarkable degree of accuracy by the Greek, Euclid. His researches on this subject, if original with him, place him among the keenest mathematicians of all time, but his work and all of his results are expressed in geometrical language which is very far removed from the algebraic symbolism of to-day.

176. Factors involving radicals. In the chapter on Factoring it was definitely stated that factors involving radicals would not then be considered. This limitation on the character of a factor is no longer necessary. Consequently many expressions which previously have been regarded as prime may now be thought of as factorable.

Thus $3x^2-1=(x\sqrt{3}+1)(x\sqrt{3}-1)$,

and $4x^2-5=(2x+\sqrt{5})(2x-\sqrt{5})$.

Therefore in this extension of our notion of a factor it must be clearly understood that the use of radicals is limited to the coefficients in the terms of the factors.

To restrict the use of radicals in the way just indicated is necessary for the sake of definiteness. Otherwise it would be impossible to obey a direction to factor even so simple an expression as x^2-y^2; for if the unknown is allowed under a radical sign in a factor, x^2-y^2 has countless factors.

Thus
$$\begin{aligned}x^2-y^2&=(x+y)(x-y)\\&=(x+y)(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})\\&=(x+y)(\sqrt{x}+\sqrt{y})(\sqrt[4]{x}+\sqrt[4]{y})(\sqrt[4]{x}-\sqrt[4]{y}),\end{aligned}$$
and so on indefinitely.

EXERCISES

Factor:

1. x^2-11.
2. $3x^2-8$.
3. x^3+3.
4. x^3-12.
5. $3x^3-27$.
6. $5x^3+125$.

Find the algebraic sum of the following:

7. $\dfrac{2\sqrt{b}}{a-b}+\dfrac{2}{\sqrt{a}+\sqrt{b}}$.
8. $\dfrac{x+c}{\sqrt{x}-\sqrt{c}}-\dfrac{x^{\frac{3}{2}}+c^{\frac{3}{2}}}{x-c}$.

Solve by factoring, and check the results:

9. $x^2-5=0$.
10. $2x^2-3=0$
11. $x^4+144=26x^2$.
12. $4x^4+c=x^2+4cx^2$.

PROBLEMS

(Obtain answers in simplest radical form.)

1. The side of an equilateral triangle is 8. Find the altitude.

2. The side of an equilateral triangle is s. Find the altitude and the area.

3. The altitude of an equilateral triangle is 24. Find one side and the area.

4. Find the side of an equilateral triangle whose altitude is a.

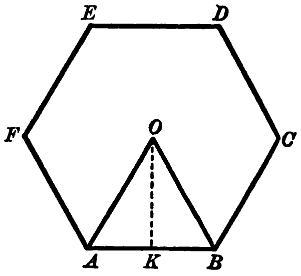
5. Find the altitude on the longest side of the triangle whose sides are 11, 13, and 20. Find the area of the triangle.

HINT. Let the altitude on the side 20 be x, and the two parts into which the altitude divides side 20 be y and $20-y$; then set up two equations involving x and y, and solve.

6. Find the altitude on the longest side of the triangle whose sides are 10, 12, and 16.

Fact from Geometry. A regular hexagon may be divided into six equal equilateral triangles by lines from its center to the vertices.

In the adjacent regular hexagon, $AB = BC = CD$, etc. O is the center and OK perpendicular to AB is the **apothem** of the hexagon.



7. Find the apothem and the area of a regular hexagon (*a*) whose side is 18; (*b*) whose side is s.

8. Find the side and the area of a regular hexagon (*a*) whose apothem is 30; (*b*) whose apothem is h.

Facts from Geometry. The volume of a pyramid or cone is $\frac{ab}{3}$, where a is the altitude and b is the area of the base.

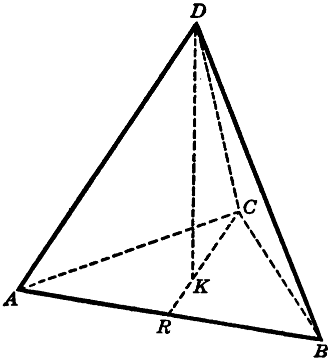
The altitudes of an equilateral triangle intersect at a point which divides each altitude into two parts whose ratio is 2 to 1.

9. The base of a pyramid is a square, each side of which is 10 feet. The other four edges are each 20 feet. Find the altitude and the volume of the pyramid.

10. The side of an equilateral triangle is 36. Find the two parts into which each altitude is divided by the other altitudes.

A **regular tetrahedron** is a pyramid whose four faces are equal equilateral triangles.

The altitude of a regular tetrahedron (*DK* in the adjacent figure) meets the base at the point where the altitudes of the base intersect.



11. *ABCD* is a regular tetrahedron. If each edge is 24, find *CR*, *CK*, and, lastly, the altitude *DK*.

12. Find the altitude and volume of a regular tetrahedron each of whose edges is 30 inches.

13. Show that the altitude and the volume of a regular tetrahedron whose edge is e are respectively $\frac{e}{3}\sqrt{6}$ and $\frac{e^3}{12}\sqrt{2}$.

MISCELLANEOUS EXERCISES

Reduce to respectively equivalent fractions having rational denominators:

1. $\frac{3\sqrt{5}}{\sqrt{3}}$.

2. $\frac{2}{3\sqrt{7}}$.

3. $\frac{2}{\sqrt{3}-\sqrt{2}}$.

4. $\frac{2\sqrt{3}+3\sqrt{5}}{2\sqrt{3}-3\sqrt{5}}$.

5. $\frac{a\sqrt{m}-b\sqrt{c}}{a\sqrt{m}+b\sqrt{c}}$.

6. $\frac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}$.

Find the positive square roots of the following:

7. $28 + 10\sqrt{3}$. **8.** $57 - 12\sqrt{15}$. **9.** $c^4 + c + 2c^2\sqrt{c}$.

10. Show that $x = \dfrac{7 + 3\sqrt{5}}{2}$ is a root of $x^2 - 7x + 1 = 0$.

11. Show that $x = \dfrac{-3 \pm 2\sqrt{3}}{3}$ are roots of $3x^2 + 6x = 1$.

12. Show that $x = \dfrac{-ab + \sqrt{60ab + a^2b^2}}{6a}$ is a root of the equation $3ax^2 + abx = 5b$.

Simplify:

13. $3\sqrt{\frac{2}{5}} + 2\sqrt{\frac{1}{10}} - 4\sqrt{\frac{5}{2}} + \sqrt{640}$.

14. $(2 + \sqrt{3}) \div \sqrt{6}$.

15. $(5\sqrt{7} - \sqrt{2}) \div (\sqrt{2} + \sqrt{7})$.

16. $3\sqrt{\frac{1}{6}} + 5\sqrt{24} + \frac{1}{3}\sqrt{96} - \sqrt{66\frac{2}{3}}$.

17. $\dfrac{\sqrt{x}}{\sqrt{x} - \sqrt{a}} - \dfrac{\sqrt{a}}{\sqrt{x} + \sqrt{a}}$.

18. $\sqrt{40} - \dfrac{1}{\sqrt{10}} + \sqrt{\dfrac{2}{5}} + 7\sqrt{\dfrac{5}{2}}$.

19. $\sqrt{(10)^2 - (2\sqrt{2})^2}$.

20. $\sqrt{1 - \left(\dfrac{\sqrt{2} - 2}{2}\right)^2}$.

21. $\sqrt{R^2 - \left(\dfrac{R}{3}\sqrt{2}\right)^2}$.

22. $\sqrt{(2R)^2 - (R - R\sqrt{2})^2}$.

23. $\sqrt{(R - R\sqrt{3})^2 - (2R - R\sqrt{2})^2}$.

24. $\sqrt{R^2 - \left(\dfrac{R}{2}\sqrt{5} - \dfrac{R}{2}\right)^2}$.

25. $\sqrt{\left(e^x + \dfrac{2}{e^x}\right)^2 - (e^x - 2e^{-x})^2 + e^{2x} + e^{-2x} - 6}$.

26. $\dfrac{b\sqrt{ax - x^2} - bx(ax - x^2)^{-\frac{1}{2}}(a - 2x)}{(\sqrt{ax - x^2})^2} \div \dfrac{bx}{\sqrt{ax - x^2}}$.

CHAPTER XXV

FUNCTIONS AND THEIR GRAPHS

177. Functions. One of the most important concepts of mathematics is the notion of **function.**

As the term is used in mathematics the basic idea is the dependence of one quantity upon another or upon several others. Countless functional relations exist in everyday affairs. The velocity of a falling body is a function of the time since it started to fall, the interest on a definite sum of money is a function of the time and the rate, and the quantity of water transported yearly by the Mississippi is a function of the rainfall (and the snowfall) in its basin. Relations such as these can often be expressed either exactly or approximately by equations. For a body falling from rest near the earth's surface, $s = 16\,t^2$. Here the distance in feet, s, is expressed as a function of the time, t, in seconds. In $y = x^2 + 3\,x + 1$ *the value of y is a function of x.* Such a relation is often represented more clearly and strikingly by a graph of the equation than by the equation itself.

178. Names of functions. A function is called **linear, quadratic,** or **cubic** according as its degree with respect to the unknown or unknowns is first, second, or third respectively.

Thus $4\,x - 7$ is a linear function of x; $3\,x^2 - 8\,x + 1$ is a quadratic function of x; and $x^3 - 3\,x^2 + x - 10$ is a cubic function of x.

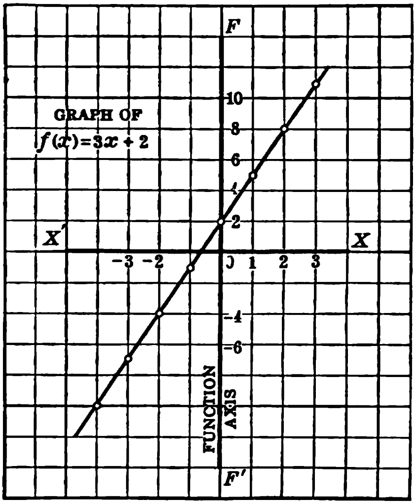
In the study of functions the unknown is often called the variable, since from this point of view the problem is not so much the finding of an unknown as it is the study of the changes of a variable quantity.

179. Notations for a function. After a function of any variable x has once been given it is usual to refer to it later in the same discussion by the symbol $f(x)$, which is read **the function of x**, or, more briefly, **f of x**.

180. Linear functions. The expression $3x + 2$ is a function of x, and the value of this binomial varies with x. The following table gives a partial view of the relative change of values between x and the function $3x + 2$:

If $x =$	-4	-3	-2	-1	0	1	2	3
then $f(x) = 3x + 2 =$	-10	-7	-4	-1	2	5	8	11

This relation can be represented graphically by using the same x-axis as before (section 91) and using the y-axis as the function axis; that is, laying off values of x horizontally and corresponding values of the function $3x + 2$ vertically. The graph resulting from the above table of values is shown in the accompanying figure. It can be shown that the graph of a linear function is always a **straight line.**



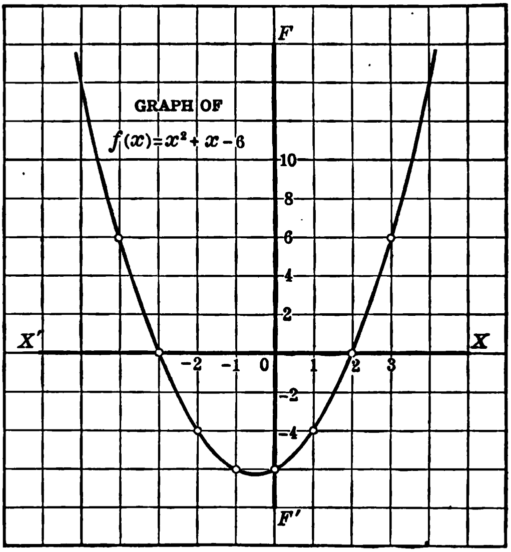
EXERCISES

1. Construct the graph of the function $3x - 1$.
2. Construct the graph of the function $2x + 8$.

181. Quadratic functions. The function x^2+x-6 may be represented graphically by proceeding as follows:

If $x=$	-4	-3	-2	-1	0	1	2	3
then $f(x)=x^2+x-6=$	6	0	-4	-6	-6	-4	0	6

Plotting the points corresponding to the numbers in the table we obtain the accompanying graph.



The graph of a quadratic function in one variable is a curve called a **parabola.** It may be sharper or flatter than the accompanying graph, but of the same general shape, and the opening may be upward or downward.

EXERCISES

Construct the graph of the following:

1. $x^2+3x-15$.
2. $3x^2+5x+2$.
3. x^2+4.
4. $2x^2-5$.
5. $15+2x-x^2$.
6. x^2+2x+1.

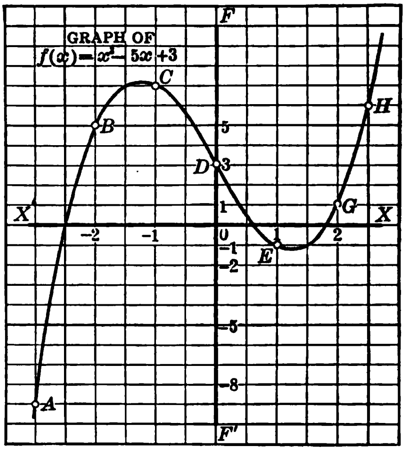
7. A body falling from rest near the earth's surface obeys the law $s=16t^2$, where s is in feet and t is in seconds. Construct the graph of $f(t)=16t^2$.

NOTE. In the study of analytical geometry one takes up systematically the curves which represent equations of the various degrees beginning with the simplest. It turns out, as we have already seen, that the linear equation is represented by a straight line. Equations of the second degree in x and y lead to the so-called "conic section."

One of the most interesting and important aspects of the graphical method is the fact that the simplest equations correspond to the most useful curves both in pure science and in nature. The commonest curves in nature are the circle and the parabola. Their equations are the very simplest equations of the second degree.

182. Graph of a cubic function. The graph of a cubic function is obtained in the same general way as that of a quadratic function. The function $x^3 - 5x + 3$ may be represented graphically by proceeding as follows:

If $x =$	-4	-3	-2	-1	0	1	2	$2\frac{1}{2}$	3
then $f(x) = x^3 - 5x + 3 =$	-41	-9	5	7	3	-1	1	$6\frac{1}{8}$	15



Plotting the points corresponding to the numbers in the table (except the first and last), we obtain the points $A(-3, -9)$, B, C, D, E, G, and H, in the order named. The curve crosses the x-axis three times: once between 1 and 2, again between 0 and 1, and a third time between -2 and -3. Above H the curve rises indefinitely, and below A it falls indefinitely. In each case it becomes more and more nearly straight as it recedes from the x-axis, never crossing either axis again.

In forming a table of values two pairs are sufficient for a linear function, but more are needed for a quadratic function and still more for a cubic function. Usually the higher the degree of the function, the more points are needed in constructing its graph. It should be noted that in making a good graph the number of points is not so important as is their distribution, which should be such as faithfully to outline the entire curve. Where the graph curves rapidly or makes sharp turns the points should be close together. Such places are difficult to locate before the graph is constructed; hence one should make a table of values which appears to be sufficient and plot them. Then inspection of the plotted points will usually show where sharp turns or rapid curvature exists. The table of values should then be properly extended and the additional points located. Repetition of this last step will enable one to draw a graph which accurately pictures the variation of the function.

It should be observed here that scales on the two axes need not be the same. Some experience is required to choose for the two axes the scales which are best suited to bring out clearly the shape of the curve. In general the graph should be drawn to *as large* a scale, *in both directions*, as the size of the paper permits. What that will be for each axis can be decided by inspecting the table of values. For example, when the dimensions of the preceding graph are once determined one can see from the table that all values of x are easily represented but that it is undesirable to try to represent the values of the function, -41 and 15.

EXERCISES

Construct the graph of the following:

1. $x^3 - 3x + 1$.
2. $x^3 - 8x + 2$.
3. $x^3 - 4x - 2$.
4. $x^4 - 11x^2 + 24$.

NOTE. The notion of a function is one of the three or four most fundamental ideas in modern mathematics. Only the simplest examples are given in this book, but many others involving expressions of the utmost complexity have been studied by mathematicians for many years. An important reason for the study of functions is found in the fact that all kinds of facts and principles which we meet in the study of nature can be expressed symbolically by means of functions and the discovery of the properties of such functions helps us to understand the meaning of the facts. A complete understanding of the laws of falling bodies, light, electricity, or sound could never be reached without the study of the mathematical functions which these phenomena suggest.

When the electrician, the architect, or the artillerist meets a problem, he frequently must represent quantities by letters. The x and the y may represent the measures of objects in nature, but the solution has become merely an operation of algebra. As students of algebra we are not concerned with the origin of the function or expression, but merely with the numerical determination of some unknown or the simplification of some expression.

183. Graphical solution of equations in one unknown. In elementary algebra one of the most important applications of the preceding sections is their use in solving equations in one unknown. The ideas involved can be made clear by questions on the graphs of sections 180, 181, and 182.

ORAL EXERCISES

1. From the graph in section 180 determine the value of the function $3x + 2$ at the point where its graph crosses the x-axis.

2. Does this value of x satisfy the equation $3x + 2 = 0$?

3. Solve $3x + 2 = 0$ without reference to the graph.

4. What point on the graph represents the root of $3x + 2 = 0$?

5. From the graph in section 181 determine the values of the function $x^2 + x - 6$ at the points where its graph crosses the x-axis.

6. Do these values of x satisfy the equation $x^2 + x - 6 = 0$?

7. Solve $x^2 + x - 6 = 0$ without referring to the graph.

8. What points on the graph represent the roots of $x^2 + x - 6 = 0$?

9. From the graph in section 182 determine the values of the function $x^3 - 5x + 3$ at the points where its graph crosses the x-axis.

10. Do these values of x make the function $x^3 - 5x + 3$ equal zero?

11. What method could be used to solve the equation $x^3 - 5x + 3 = 0$?

184. The process of graphical solution. From what precedes, it is apparent that the steps in the graphical solution of an equation in one unknown are:

Transpose the terms so that the right member is zero.

Graph the function in the left member.

The values of x for the points where the graph crosses the x-axis are the real roots of the equation.

The algebraic solutions of a linear equation and a quadratic equation in one unknown are so simple that except for the purpose of illustration their graphical solution is comparatively unimportant. The algebraic solution of cubic and higher equations, except in simple cases, is much more difficult than the solution of the quadratic and is never presented in an elementary course. For this reason and for the insight it gives into equations in general, the *graphical* solution of the cubic and higher equations is important and illuminating. For such equations if the method of factoring fails, the graphical method is the only method open to the student at this point in his progress.

EXERCISES

Solve graphically:

1. $x^2 - 7x + 3 = 0.$

2. $x^3 - 5x + 4 = 0.$

3. $x^3 - 3x + 1 = 0.$

4. $x^3 - 8x = 0.$

5. $x^3 - x^2 + 5 = 0.$

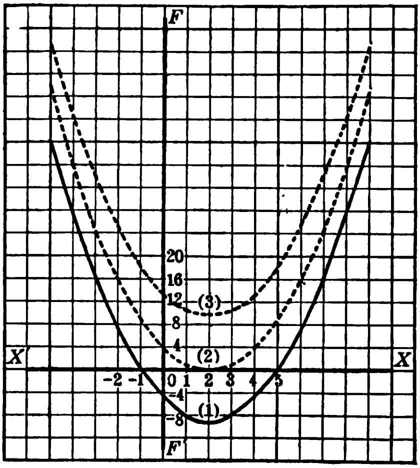
6. $x^4 - 10x^2 + 16 = 0.$

185. Imaginary roots. An equation of the second or a higher degree often has **imaginary roots.** Such roots cannot be obtained by the graphical methods so far considered. A study of the graphs which follow will make clear why this is true.

Consider the following equations:

$$x^2-4x-5=0, \tag{1}$$
$$x^2-4x+4=0, \tag{2}$$
$$x^2-4x+13=0. \tag{3}$$

The graphs of the functions in the left members of equations (1), (2), and (3) are given in the accompanying figure. The three functions differ only in their constant terms, for 9 added to the constant term of (1) gives the constant term of (2), and 9 added to the constant term of (2) gives the constant term of (3). Apparently, as the constant term is increased the graph rises without change of shape and without motion to the left or to the right.



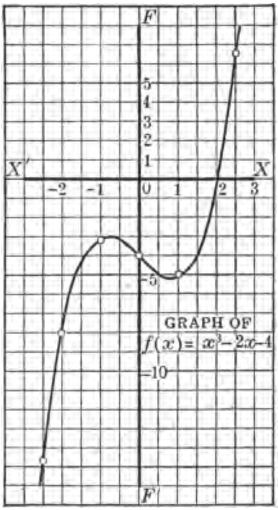
From the graph the roots of $x^2-4x-5=0$ are seen to be 5 and -1. These results are obtained from factoring; $x^2-4x-5=0$, or $(x-5)(x+1)=0$. Whence $x=5$ or -1.

If we imagine curve (1) to move upward, the two roots change in value and become the single root of curve (2), which touches the x-axis at a point where x equals 2. Solving $x^2 - 4x + 4 = 0$ by factoring gives $(x-2)(x-2) = 0$. Whence $x = 2$.

If we now imagine curve (1) to move still farther upward from its position (2), it will no longer cut the x-axis. Further, when the curve reaches the position of (3) it does not cut the x-axis at all, and hence cannot show the values of the roots of the equation $x^2 - 4x + 13 = 0$, as, in fact, it does not. The graph does show, however, that the value of $x^2 - 4x + 13$ at the lowest point of the curve is 9. This means that for every real value of x, positive or negative, $x^2 - 4x + 13$ is never less than 9. The graph of (3) makes clear that no real number if substituted for x will make $x^2 - 4x + 13$ equal zero.

It can be shown by the method of section 87 that the roots of $x^2 - 4x + 13 = 0$ are $2 + 3\sqrt{-1}$ and $2 - 3\sqrt{-1}$.

NOTE. It required the genius of no less a man than Sir Isaac Newton first to observe from the graph of a function that two of its roots become imaginary simultaneously. He also saw that an equation with two of its roots equal to each other is, in a certain sense, the limiting case between equations in which the corresponding roots appear as two real and distinct roots and those in which they appear as imaginary roots.



186. Imaginary roots for a cubic equation. If we attempt to solve $x^3 - 2x - 4 = 0$ graphically, we obtain the graph of the accompanying figure. The curve crosses the x-axis at $x = 2$. This is the only real root the equation has; the other two are imaginary. The roots can here be obtained by factoring $x^3-2x-4=0$; thus $(x-2)(x^2+2x+2)=0$. The roots of $x^2+2x-2=0$ are the two imaginary roots of the *cubic* equation.

EXERCISES

As far as possible solve graphically, finding results to one decimal place:

1. $x^2-5x-9=0$.
2. $x^2=4x-5$.
3. $x^3-3x+4=0$.
4. $x^3+x-4=0$.
5. $x^3-4x^2-2x+8=0$.
6. $x^3-4x+5=0$.
7. $x^3-7x+4=0$.
8. $x^4-4x^3+12=0$.
9. $x^4=10x^2-9$.
10. $x^4-2x^2=0$.

CHAPTER XXVI

QUADRATIC EQUATIONS

(In Part Review)

187. Solution by completing the square. The quadratic equation is defined on page 137. The method of solving a quadratic equation by completing the square is explained and illustrated on pages 270–271.

EXERCISES

Solve by completing the square and check the results as directed by the teacher:

1. $x^2 + 4x + 3 = 0.$	**5.** $3x^2 + 7x - 6 = 0.$
2. $s^2 - s - 2 = 0.$	**6.** $2x^2 - 3x - 5 = 0.$
3. $x^2 + 2 = -3x.$	**7.** $5x^2 - 7x - 6 = 0.$
4. $2x^2 + 5x + 3 = 0.$	**8.** $x - 4 + x^2 = 6 - 2x^2 + 8x.$

In Exercises 9–13 obtain results to two decimal places:

9. $x^2 - 12x + 31 = 0.$

Solution. By applying the rule we get

$$x = 6 + \sqrt{5},$$

and

$$x = 6 - \sqrt{5}.$$

From the table on page 502, $\sqrt{5} = 2.236.$

Hence we get the result, to two decimals,

$$x = 8.23 \text{ and } 3.76.$$

10. $n^2 + 2 = 5n.$	**12.** $5x^2 + 8x + 2 = 0.$
11. $3x^2 - 12x + 9 = 2.$	**13.** $(y+1)(y+2) = 3y(y-4).$

14. $x^4 - 3x^2 + 2 = 0.$

NOTE. This is not a quadratic equation, but many equations of this form can be solved by the methods applicable to quadratics.

Solution. $x^4 - 3x^2 + 2 = 0.$

$$x^4 - 3x^2 = -2.$$

$$x^4 - 3x^2 + \tfrac{9}{4} = -2 + \tfrac{9}{4} = \tfrac{1}{4}.$$

$$(x^2 - \tfrac{3}{2})^2 = \tfrac{1}{4}.$$

$$x^2 - \tfrac{3}{2} = \pm \tfrac{1}{2}.$$

$$x^2 = 2 \text{ or } 1.$$

Whence $x = \pm\sqrt{2}, \pm 1.$

Check as usual.

NOTE. It should be particularly observed that the equation of Exercise 14 has four roots instead of two. In general an equation has a number of roots equal to its degree.

15. $x^4 - 5x^2 + 4 = 0.$

16. $x^4 - 13x^2 + 36 = 0.$

17. $x^4 - 4x^2 + 3 = 0.$

18. $6x^4 - 11x^2 + 3 = 0.$

19. $x^{-2} + 16x^{-1} - 17 = 0.$

20. $y^{-4} - 10y^{-2} + 9 = 0.$

21. $x^{-1} - 13x^{-\frac{1}{2}} = -36.$

22. $(x^2 - 2x)^2 - 7(x^2 - 2x) = -12.$

Solution. Let $x^2 - 2x = y.$

Substituting y for $x^2 - 2x$, we obtain

$$y^2 - 7y = -12.$$

Solving, $y = 3$ and $4.$

Then $x^2 - 2x = 3.$

Whence $x = 3$ and $-1.$

Also $x^2 - 2x = 4.$

Whence $x = 1 \pm \sqrt{5}.$

In Exercises 23–25 do not expand, but solve as in Exercise 22:

23. $(x - 1)^2 + 4(x - 1) = 5.$

24. $(x^2 - 4x)^2 - 5(x^2 - 4x) - 24 = 0.$

25. $\left(x-\frac{6}{x}\right)^2+4\left(x-\frac{6}{x}\right)-5=0.$

26. $ax^2+bx+c=0.$

Solution. $ax^2+bx+c=0.$

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}.$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

27. $x^2+3ax+2a^2=0.$

28. $2y^2+by=6b^2.$

29. $bx^2+x=1+bx.$

30. $\frac{x-c}{c}-\frac{c}{x-c}=\frac{3}{2}.$

31. $bx^2-a(b+1)x+a^2=0.$

32. $\frac{b}{2a}x^2=\frac{b}{2a}+1-x.$

33. $cx^2-(c^2+d)x+cd=0.$

34. $abx^2-x(a+b)\sqrt{ab}+ab=0.$

35. $6a^2y^2-7ary+2r^2=0.$

36. $(ax+b)^2+3(ax+b)+2=0.$

188. Solution by formula. In Exercise 26, above, the general quadratic $ax^2+bx+c=0$ has been solved and the roots found to be

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \qquad (F)$$

The expression (F) is a general result and may be used as a formula to solve any quadratic equation in the standard form $ax^2+bx+c=0$, in which a, b, and c may represent numbers, single letters, binomials, or any other form of algebraic expression not involving x.

If the numbers a, b, and c are such that the expression b^2-4ac is negative, the formula contains the square root of a negative number, which is a kind of number not yet fully considered in this text.

In the exercises that follow it will be assumed that only such numerical values of the literal coefficients are involved as will not make $b^2 - 4ac$ negative. A discussion of the case here ruled out will be found in Chapter XXVII.

EXERCISES

Solve for x by formula and check the results as directed by the teacher:

1. $4x^2 + 8x = 3$.

Solution. Writing in standard form,

$$4x^2 + 8x - 3 = 0.$$

Comparing with $ax^2 + bx + c = 0$, evidently 4 corresponds to a, 8 to b, and -3 to c. Substituting the values in (F) gives

$$x = \frac{-8 \pm \sqrt{64 - 4 \cdot 4 \cdot (-3)}}{2 \cdot 4}$$

$$= \frac{-8 \pm \sqrt{64 + 48}}{8} = \frac{-8 \pm \sqrt{112}}{8}$$

$$= \frac{-8 \pm 4\sqrt{7}}{8} = -1 \pm \tfrac{1}{2}\sqrt{7}.$$

Check as usual.

[illegible] $x^2 - 6x - 16 = 0$

[illegible] $x^2 - x - 1 = 0$.

[illegible] $3x^2 + 7x - 3 = 0$.

[illegible] $2x + 4 = x^2$.

6. $x = 1 - x^2$.

~~**7.** $3x + 4 = x^2$.~~

8. $11x^2 - 15x = 26$.

~~**9.** $\frac{3}{2}x - \frac{1}{2} = x^2$.~~

10. $3p^2x^2 = px + 4$.

Solution. Writing in standard form,

$$3p^2x^2 - px - 4 = 0.$$

Here $a = 3p^2$, $b = -p$, and $c = -4$.

Substituting these values in the formula (F),

$$x = \frac{-(-p) \pm \sqrt{(-p)^2 - 4 \cdot 3p^2(-4)}}{2 \cdot 3p^2}$$

$$= \frac{p \pm \sqrt{p^2 + 48p^2}}{6p^2} = \frac{p \pm 7p}{6p^2} = \frac{4}{3p} \text{ and } \frac{-1}{p}.$$

Check as usual.

11. $x^2 - 2kx - 3k^2 = 0.$

12. $3a^2 = 7ax + 6x^2.$

13. $4x^2 + kx - 14k^2 = 0.$

14. $bx = 12x^2 - b^2.$

15. $a^2x^2 + 4abx + 3b^2 = 0.$

16. $12p^2x^2 - 4prx - r^2 = 0.$

17. $3a^2x^2 + 8abx + 4b^2 = 0.$

18. $x - 3\sqrt{c} - \dfrac{4c}{x} = 0.$

19. $6h^2 + \dfrac{13hn}{x} = \dfrac{8n^2}{x^2}.$

20. $13a^2b^2x^2 = 9b^4 + 4a^4x^4.$

21. $2x^2 + 5x = cx^2 + 3cx + 3.$

Solution. $(2 - c)x^2 + (5 - 3c)x - 3 = 0.$

Here $a = 2 - c, \quad b = 5 - 3c, \quad$ and $\quad c = -3.$

Hence
$$x = \frac{3c - 5 \pm \sqrt{(5 - 3c)^2 + 12(2 - c)}}{2(2 - c)}$$
$$= \frac{3c - 5 \pm \sqrt{49 - 42c + 9c^2}}{4 - 2c}$$
$$= \frac{3c - 5 \pm (7 - 3c)}{4 - 2c}$$
$$= \frac{2}{4 - 2c}, \text{ or } \frac{1}{2 - c}, \text{ and } \frac{6c - 12}{4 - 2c}, \text{ or } -3.$$

22. $2x^2 + (a + 2b)x = a^2 + ab.$

23. $ax^2 - 2ax - 3a = bx - \frac{b}{3}x^2.$

24. $p^2 + x^2 = 2px + 2x - 2p.$

25. $x^2 - cx = \frac{1}{2}(ax - ac).$

26. $4a^2x^4 + 4c^2 = 16a^2x^2 + c^2x^2.$

27. $x^6 - a^8x^3 = 8b^3x^3 - 8a^3b^3.$

28. $a^2 + x^2 = 2ax + b^2.$

29. $12pq = x^2 + 4qx - 3px.$

30. $x^4 - a^2x^2 + a^2b^4 = b^4x^2.$

189. Comparison of the various methods. Four methods have been given for the solution of the quadratic equation:

(*a*) Solution by factoring.
(*b*) Solution by graphing.
(*c*) Solution by completing the square.
(*d*) Solution by formula.

In practice, the first and the last of these methods are most convenient.

When an equation with integral coefficients can be solved by factoring, as in the examples on pages 139–140, the roots are always rational numbers.

For example, $3x^2 + 2x - 8 = 0$ factors into $(3x - 4)(x + 2) = 0$. Hence the roots are $x = \frac{4}{3}$ and -2.

An inspection of the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

shows that if a, b, and c are integers, one always obtains roots involving radicals unless the expression under the radical sign, $b^2 - 4ac$, is a perfect square. In this case, however, the values of the roots are rational numbers.

For example, in the equation $3x^2 + 2x - 8 = 0$ the value of the expression $b^2 - 4ac$ is $(2)^2 - 4 \cdot 3(-8) = 100$, which is a perfect square. Hence the roots of $3x^2 + 2x - 8 = 0$ are rational numbers, since the radical term in the roots can be expressed as a rational number.

Hence to determine whether a quadratic equation of the form $ax^2 + bx + c = 0$ can be solved after factoring into rational factors, we have the

Rule. *Compute the value of $b^2 - 4ac$ for the equation.*

If the result is the square of an integer, the left member of the equation can be factored and the roots are rational.

ORAL EXERCISES

Determine which of the following can be solved by factoring:

1. $x^2 + 10x + 25 = 0.$

2. $x^2 - 5x - 400 = 0.$

3. $x^2 - 7x - 3 = 0.$

4. $2x^2 + 3x + 1 = 0.$

5. $2x^2 + 3x + 2 = 0.$

6. $x^2 - 4x - 4 = 0.$

7. $3x^2 - x + 4 = 0.$

8. $3x^2 - 5x + 8 = 0.$

9. $2x^2 - 10x - 25 = 0.$

10. $3x^2 + 2ax - a^2 = 0.$

REVIEW EXERCISES

Solve the following for x by the method best adapted to each:

1. $3x^2 - 7x - 10 = 0.$

2. $5x^2 + 14x + 8 = 0.$

3. $4x^2 + 3x - 7 = 0.$

4. $x^2 - 1.7x - .84 = 0.$

5. $8x^2 + 2\sqrt{5}x - 15 = 0.$

6. $x^4 - 7x^2 + 12 = 0.$

7. $x^3 - x = 0.$

8. $x^4 = x^2.$

9. $x^4 - 64x^2 = 0.$

10. $(x^2 - 3x)^2 - 2(x^2 - 3x) = 8.$

11. $5x^2 - 9x + 3 = 0.$

~~12.~~ $.03x^2 + .01x = .1.$

13. $x^2 + x\sqrt{2} - \sqrt{6} = x\sqrt{3}.$

14. $\dfrac{x-1}{x} = \dfrac{x+1}{6}.$

15. $\dfrac{x-3}{x+4} = \dfrac{x-4}{2x+6}.$

16. $\dfrac{x+1}{x+2} + \dfrac{x+3}{x+4} = \dfrac{22}{15}.$

17. $\dfrac{2x-1}{x-1} + \dfrac{1-x}{x-2} = \dfrac{x+1}{x+2}.$

18. $pqx^2 - rqx + psx = rs.$

19. $x^2 - 2ax + a^2 - b^2 = 0.$

20. $\dfrac{m}{m-x} - \dfrac{m-x}{m} = 2.$

21. $3x^2 + 2a = x + 6ax.$

22. $5x^2 + 3m = 5mx + 3x.$

23. $ax^2 - hm = mx - ahx.$

24. $cx^2 + kx = 2mk + 2cmx.$

25. $x^2 + 2s + s^2 = 2sx + 2x.$

26. $c^2x^2 + a^2 = m^2 + 2acx.$

27. $cmx^2 + m^2x + a^2 = am + acx + amx.$

28. $ax^2 - 4a = bx^2 + 2bx.$

29. $2mx^2 + 3ax = 18m + ax^2.$

30. $(s - x)^2 + (t - x)^2 = (t - s)^2.$

31. $3cx^2 + 50a^2 = 15acx + 2x^2.$

32. $2x^2 + 5x = cx^2 + 3cx + 3.$

33. $x^2 + 2x + 1 = mx^2 + mx.$

34. $ax^2 + 3x = 2x^2 + 2ax - 2.$

3

35. $2x^2 + 5x = hx^2 + 3hx + 3.$

36. $2x^2 + (a + 2b)x = a^2 + ab.$

37. $(3a + b)x^2 - (2a + b)x = a.$

38. $\dfrac{t^2}{x^2} = \dfrac{t+1}{x+1}.$

39. $\dfrac{x + 4a + c}{c - a} = \dfrac{9c + x}{x}.$

40. $\dfrac{1}{mx + 4} + \dfrac{mx - 4}{16} = 1.$

41. $\dfrac{2a + x}{2a - x} + \dfrac{a - 2x}{a + 2x} = \dfrac{8}{3}.$

42. $\dfrac{x^2 + 1}{x} = \dfrac{a + m}{t} + \dfrac{t}{a + m}.$

43. $\dfrac{2x - r}{s} - \dfrac{4r}{2x - s} + 3 = 0.$

44. $\dfrac{1}{m + c + x} = \dfrac{1}{m} + \dfrac{1}{c} + \dfrac{1}{x}.$

45. $\dfrac{x + a}{x - t} + \dfrac{x - a}{x + t} = 1 + \dfrac{a^2 + 2t^2}{x^2 - t^2}.$

46. $\dfrac{x^2 + 2ax}{a - b} - \dfrac{x}{2b} = \dfrac{a}{b}.$

47. $(x^2 + 5x + 2)^2 - 6(x^2 + 5x + 2) = 16.$

48. $\dfrac{3x - 1}{5x + 1} - \dfrac{x + 1}{x - 1} = \dfrac{48}{(5x + 1)(1 - x)}.$

49. $x - 3 = \dfrac{x^3 - 10x^2 + 1}{(x - 3)^2}.$

50. $15x^2 - 1.95x + .054 = 0.$

51. $10 - 9x = 7x^2.$

52. $x^2 + 961a^2 = 62ax.$

53. $(2x + 3)(x - 4) = (3x - 8)(4x - 1).$

54. $(3x - 2)(x - 5) = (4x - 3)(x + 1).$

55. $\dfrac{1}{x+7}=\dfrac{2}{5}-\dfrac{1}{3-x}.$

56. $\dfrac{x}{x^2+3x-4}=\dfrac{x+2}{x+4}.$

57. $\dfrac{x+3}{x^2-3x+2}=\dfrac{x-1}{x-2}-\dfrac{x-2}{x-1}.$

58. $\dfrac{5x-11}{x-1}+\dfrac{1-3x}{x+2}=\dfrac{2x-1}{x+1}.$

59. $17x=6x^2-10.$

60. $2k^2x^2+8kx=5(3kx-1).$

61. $37pqx=210q^2-p^2x^2.$

62. $\dfrac{3x-2}{x-1}+\dfrac{2}{3}=9\tfrac{1}{6}-\dfrac{5(x+1)}{4x-1}.$

63. $3+\dfrac{4k}{c-2x}=\dfrac{k-2x}{c}.$

~~64. $.3x^2$~~ $+ .7x = .2.$

also ——
Determine values of m for which the equation
$x^2+(m+5)x$
$+5m+1$ [illegible]

65. $\dfrac{1}{x^2-3x+2}+\dfrac{1}{x^2+x-6}=\dfrac{1}{3-2x-x^2}.$

66. $\dfrac{2(x^2-2)-1}{x^2-1}=\dfrac{x^2+2}{(x^2-2)-4}.$

67. $(x-2)^2-(1-2x)(3x+5)=5-(1-2x)(3x+2).$

68. $(x^2-2x-2)^2+2(x^2-2x-2)-3=0.$

69. $(x^2-4x+1)^2-4(x^2-4x)-16=0.$

70. $(2x-b)^2=a(2x-b)+2a^2.$

PROBLEMS

1. Separate 42 into two parts such that the first shall be the square of the second.

2. Find two consecutive odd numbers whose product is 143.

3. The sum of the reciprocals of two consecutive numbers is $\frac{13}{42}$. Find the numbers.

4. The time, in hours, required for a trip of 216 miles was 6 greater than the rate, in miles per hour. Find the time and the rate.

5. The altitude of a triangle is 6 feet less than the base. The area is 56 square feet. Find the base and altitude.

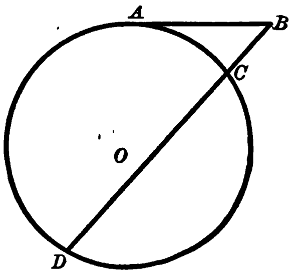
6. One leg of a right triangle is 9 feet shorter than the other, and the area is 45 square feet. Find the three sides.

7. The area of a trapezoid is 180 square feet. One base is 4 feet greater than the altitude, the other is three times the altitude. Find the two bases and the altitude.

8. A rectangle whose area is 27 square inches has a perimeter of 21 inches. Find the length of the shorter side.

9. A polygon of n sides always has $\frac{1}{2}n(n-3)$ diagonals. How many sides has a polygon with 119 diagonals?

10. If AB, in the accompanying figure, is a tangent to the circle and BD is any secant, then $(AB)^2 = BC \cdot BD$. If $AB = 6$ and $CD = 10$, find BC.



11. A requires 4 more days than B to do a piece of work. Working together they require $2\frac{2}{3}$ days. How many days will each require alone?

12. In selling an article at $20, a merchant's per cent of profit is 9 greater than the original cost of the article in dollars. Find the original cost and the per cent of profit.

13. A 3-inch square is cut from each corner of a square piece of tin. The sides are then turned up, forming an open box of volume 12 cubic inches. What was the side of the original square?

14. The number of lines on a certain printed page is 20 less than the average number of letters per line. If the number of letters per line is decreased by 16, the number of lines must be increased by 15 in order that the new page may contain as much matter as the old. How many lines and how many letters were there on the original page?

15. For what positive value or values of x will the product of $x - 5$ and $x + 5$ be 1 greater than their difference?

16. What values of x will make the product of $4x - 5$ and $x - 2$ equal in value to $x + 4$?

17. A sum of $4000 is invested, and at the end of each year the year's interest, plus $400, is added to the investment. At the beginning of the third year the investment amounts to $5230. What is the rate of interest?

18. The cost of an outing was $60. If there had been two more in the party, the share of each would have been a dollar less. How many were there?

19. The dimensions of a rectangular box are expressed by three consecutive numbers. Its surface is 214 square inches. Find its dimensions.

20. The radius of a circle is 28 inches. How much must this be shortened in order to decrease the area of the circle by 1078 square inches?

21. Two bodies, A and B, move on the sides of a right triangle. A is now 5 feet from the vertex, and moving from it at the rate of 10 feet per second. B is 35 feet from the vertex, and moving toward it at 5 feet per second. At what time (past or future) are they 75 feet apart?

22. How high is a mountain which can just be seen from a point on the surface of the sea, 40 miles distant?

HINT. See Exercise 10. Consider the radius of the earth as 4000 miles

23. The distance in feet, s, through which a body falls from rest in t seconds is given by the formula $s = \frac{1}{2}gt^2$, where $g = 32$, approximately. A bomb dropped from an aëroplane struck the ground below 8 seconds later. How high was the aëroplane at the time?

24. If a stone be thrown vertically upward, with a velocity of 100 feet per second, it is known that, neglecting the resistance of the air, its height after t seconds will be $100t - 16t^2$ feet. After how many seconds will it be 136 feet high? Explain the double answer.

25. After how many seconds will the stone of Exercise 24 return to its starting point? Explain the zero root.

CHAPTER XXVII

IRRATIONAL EQUATIONS

190. Definitions and discussion. An **irrational** equation in one unknown is an equation in which the unknown occurs under a radical, or is affected by a fractional exponent.

Thus $3x + 5\sqrt{x} = 1$, $x - x^{\frac{1}{2}} + 1 = 0$, and $\sqrt[3]{x^2 - 3x + 2} = 6$ are irrational equations.

The chief difficulty involved in the solution of such equations arises from the fact that sometimes results are obtained which do not satisfy the given equation and hence are not roots of that equation. A result of this kind is called **extraneous**.

EXAMPLE

	(a) Solve $\sqrt{x-6} - 4 = 0$.		(b) Solve $-\sqrt{x-6} - 4 = 0$.	
Solution. Transposing,				
	$\sqrt{x-6} = 4.$	(1)	$-\sqrt{x-6} = 4.$	(1)
Squaring,	$x - 6 = 16.$	(2)	$x - 6 = 16.$	(2)
Solving,	$x = 22.$		$x = 22.$	
Check.	$\sqrt{22-6} - 4 = 0.$		$-\sqrt{22-6} - 4 = 0.$	
	$\sqrt{16} - 4 = 0.$		$-\sqrt{16} - 4 = 0.$	
	$4 - 4 = 0,$		$-4 - 4 = 0,$	
	which is true.		which is not true.	

It appears from a study of these solutions that statements (1) differ only in the signs preceding their left members. Consequently this distinction disappears after squaring, and equations (2) are identical. Since the remainder of the work in both (a) and (b) consists in the solution of (2), the result obtained is really the root of this equation. Whether the root obtained satisfies both (a) and (b), or only one of them, can be determined only by substitution. Here it appears that (a) is an equation and that (b) is not, but is merely a false statement in the form of an equation.

In any case, all of the roots of the original equation are sure to be among the results found, provided no factor containing the unknown has been divided out. But no result should be called a root unless it satisfies the original equation. This means that all results must be checked.

In irrational equations, as in all the work up to the present, it is understood that unless a radical or an expression affected by a fractional exponent is preceded by the double sign $\pm$ it has only the one value, just like any other number symbol.

Thus $\sqrt{16}$ means $+4$, and not -4.

Also $4^{\frac{1}{2}}$ means $+2$, while $-4^{\frac{1}{2}}$ means $-\sqrt{4}$, or -2.

If this fact is kept in mind, it is clear from an inspection of (b), above, that it could have no root, since the sum of two negative numbers could not possibly be zero.

The method of solving equations containing radicals is stated in the

Rule. *Transpose the terms so that one radical expression (the least simple one if there are two or more) is the only term in one member of the equation.*

Next raise both members of the resulting equation to the same power as the index of this radical.

Combine like terms in each member and, if radical expressions still remain, repeat the two preceding operations *until an equation is obtained which is free from radicals; then solve this equation.*

Check. *Substitute in the original equation the values found and reduce the resulting numerical equation to its simplest form by extracting roots but not by raising both members of the equation to a higher power.*

Finally, reject all extraneous roots.

EXERCISES

Solve the following for real roots and check the results:

1. $\sqrt{x+1}=5.$

2. $\sqrt{x+1}=\sqrt{3x-5}.$

3. $3\sqrt{2x-8}-7=17.$

4. $3\sqrt{2x+6}=\sqrt{6x^2-6}.$

5. $\sqrt[3]{2x+3}=3.$

6. $7+2\sqrt[3]{2x-1}=13.$

7. $\sqrt[3]{4x+3}-\sqrt[3]{4-3x}=0.$

8. $\sqrt[3]{2x+2}=\sqrt{2x-2}.$

9. $\sqrt[4]{x+1}=\sqrt{x-1}.$

10. $1+\sqrt{x-2}=\sqrt{x}.$

Solution. Transposing, $\sqrt{x-2}=\sqrt{x}-1.$

Squaring, $x-2=x-2\sqrt{x}+1.$

Transposing and collecting, $-3=-2\sqrt{x}.$

Squaring, $9=4x$, or $x=\frac{9}{4}.$

Check. Substituting $\frac{9}{4}$ for x in the original equation,

$$1+\sqrt{\tfrac{9}{4}-2}=\sqrt{\tfrac{9}{4}},$$

$$1+\tfrac{1}{2}=\tfrac{3}{2}.$$

11. $\sqrt{9x}-4=\sqrt{x}.$

12. $\sqrt{3x-2}+5=\sqrt{x+35}.$

13. $\sqrt{3x+4}-\sqrt{x+5}=\sqrt{5-x}.$

14. $\sqrt{x+2}=\sqrt{x}+\sqrt{2}.$

15. $x+1=\sqrt{2x^2+3x-1}.$

16. $\sqrt{x+2}+\sqrt{x-1}=\sqrt{3x+3}.$

17. $x^{\frac{4}{3}} - 9x^{\frac{2}{3}} + 20 = 0.$

NOTE. The equation here given is not a quadratic equation, but it is of the general type $ax^{2n} + bx^{n} + c = 0$. Here x occurs in but two terms, and its exponent in one term is twice that in the other term. Many equations in this form can be solved by completing the square.

Solution.
$$x^{\frac{4}{3}} - 9x^{\frac{2}{3}} + \tfrac{81}{4} = -20 + \tfrac{81}{4}.$$
$$x^{\frac{2}{3}} - \tfrac{9}{2} = \pm \tfrac{1}{2}.$$
$$x^{\frac{2}{3}} = 5 \text{ or } 4.$$

Whence
$$x = \pm 5^{\frac{3}{2}} \text{ or } \pm 8.$$

Check. Substituting $\pm 5^{\frac{3}{2}}$ for x in the original equation,
$$(\pm 5^{\frac{3}{2}})^{\frac{4}{3}} - 9(\pm 5^{\frac{3}{2}})^{\frac{2}{3}} + 20 = 0,$$
or
$$5^2 - 9 \cdot 5 + 20 = 0, \text{ or } 0 = 0.$$

Substituting ± 8 for x in the original equation,
$$(\pm 8)^{\frac{4}{3}} - 9(\pm 8)^{\frac{2}{3}} + 20 = 0.$$
$-4 = 0$
$$16 - 36 + 20 = 0, \text{ or } 0 = 0.$$

4 + 5·4

18. $x^{3} + 5x^{\frac{3}{2}} - 14 = 0.$

19. $6 = x^{\frac{1}{2}} + x.$

14

20. $x^{-4} - 17x^{-2} + 52 = 0.$

21. $2x^{\frac{2}{3}} - 7\sqrt[3]{x} + 6 = 0.$

22. $x^{-\frac{3}{2}} = 4x^{-\frac{3}{4}} + 32.$

23. $x^{\frac{1}{3}} - x^{\frac{1}{6}} - 6 = 0.$

24. $x^{-1} - \dfrac{25}{36x^{\frac{1}{2}}} + \dfrac{1}{9} = 0.$

25. $x^2 + 5x + 3\sqrt{x^2 + 5x} - 54 = 0.$

HINT. Let $y = x^2 + 5x$.

26. $3x^2 - 4x - 11\sqrt{3x^2 - 4x} + 18 = 0.$

27. $2x^2 - 3x - 4 - \sqrt{2x^2 - 3x - 1} + 1 = 0.$

28. $x^2 - 2x - 5\sqrt{x^2 - 2x - 4} + 2 = 0.$

29. $12x^{\frac{4}{3}} - 27x^{\frac{2}{3}} = 20x^{\frac{2}{3}} - 45.$

algebra is a [illegible]
not to be [illegible]

30. $15 - 2\sqrt{x^2+9} = x^2 + 9.$ **31.** $\frac{6\sqrt{x}-8}{\sqrt{2x}-\sqrt{8}} = \sqrt{2}.$

32. $\frac{2\sqrt{3}}{\sqrt{2x-3}} = \frac{\sqrt{2x+12}}{3\sqrt{3}}.$ [illegible]

33. $4(x-5) - 2(x-5)^{\frac{1}{2}} - 2 = 0.$

34. $4x\sqrt{x} = (x+3)\sqrt{9x}.$ $4x(x^{1/2} = x + 3(9x)^{1/2}$

35. $\frac{\sqrt{x+2}}{\sqrt{8-x}} - \frac{\sqrt{8-x}}{\sqrt{x+2}} = \frac{8}{3}.$

36. If $t = \pi\sqrt{\frac{l}{g}}$, solve for l and g.

37. If $r = \frac{1}{2}x\sqrt{3}$ and $A = 2\pi r^2$, express A in terms of x.

38. If $a^2 = 2r^2$ and $C = 2\pi r$, express C in terms of a.

39. If $A = \frac{3r^2}{2}\sqrt{3}$ and $x = \frac{1}{2}r\sqrt{3}$, express A in terms of x.

40. If $K = 3r^2$ and $a = \frac{1}{2}r\sqrt{3+\sqrt{2}}$, express K in terms of a.

$3(x+2) - 2(x - $ [illegible]

x^3

$(x^{1/3}$ [illegible]

x

$2^3 = 1$

CHAPTER XXVIII

IMAGINARIES

191. Definitions. When the square root of a negative number arose in our previous work, it was called an imaginary, but no attempt was then made to use it or to explain its meaning. The treatment of imaginaries was deferred because there were so many topics of more importance to the beginner. It must not be supposed, however, that imaginaries are not of great value in mathematics. They are frequently used in certain branches of applied science; and it is unfortunate that symbols which can be employed in numerical computations to obtain practical results should ever have been called imaginary. By such a name something unreal and fanciful is suggested, to obviate which it has been proposed to call imaginary numbers *orthotomic numbers*, but this name has been little used.

The equation $x^2+1=0$, or $x^2=-1$, states that x is a number whose square is -1. By defining a new number, $\sqrt{-1}$, as one whose square is -1, we obtain one root for the equation $x^2+1=0$.

Similarly, $\sqrt{-5}$ is a number whose square is -5. And, in general, $\sqrt{-n}$ is a number whose square is $-n$. Obviously, $\sqrt{-5}$ means something very different from $\sqrt{5}$, and $\sqrt{-n}$ from $\sqrt{n}$.

The positive numbers are all multiples of the unit $+1$, and the negative numbers are all multiples of the unit -1. Similarly, pure imaginary numbers are real multiples of the imaginary unit $\sqrt{-1}$, as $2\sqrt{-1}$, $-5\sqrt{-1}$, and $b\sqrt{-1}$.

Furthermore $\sqrt{-4} = 2\sqrt{-1}$; $\sqrt{-a^2} = a\sqrt{-1}$; and $\sqrt{-5} = \sqrt{5} \cdot \sqrt{-1}$.

The imaginary unit $\sqrt{-1}$ is often denoted by the letter i; that is, $3\sqrt{-1} = 3i$.

If a real number be united to a pure imaginary by a plus sign or a minus sign, the expression thus obtained is called a **complex number.**

Thus $-2 + \sqrt{-1}$ and $3 - 2\sqrt{-4}$ are complex numbers. The general form of a complex number is $a + bi$, in which a and b may be any real numbers.

NOTE. Up to the time of Gauss (1777–1855) complex numbers were not clearly understood and were usually thought of as absurd. The situation reminds one of the time when negative numbers were similarly regarded, and the veil was removed from both in about the same way. It was found that negative numbers really had a significance — that they could be used in problems that involve debt, opposite directions, and many other everyday relations. The interpretation of imaginary numbers is not quite so obvious, and is not considered in this text. But as soon as it was seen that an interpretation was possible the ice was broken, and it needed only the insight and authority of a man like Gauss to give complex numbers their proper place in mathematics.

ORAL EXERCISES

Express as multiples of $\sqrt{-1}$, or i:

1. $\sqrt{-16}$.
2. $\sqrt{-25}$.
3. $\sqrt{-a^2}$.
4. $2\sqrt{-3}$.
5. $3\sqrt{-6}$.
6. $2\sqrt{-c}$.
7. $\sqrt{3} \cdot \sqrt{-3}$.
8. $\sqrt{2} \cdot \sqrt{-5}$.
9. $\sqrt{5} \cdot \sqrt{-20}$.
10. $a\sqrt{-c}$.
11. $\sqrt{-x^2 - 2x - 1}$.
12. $\sqrt{-y^2 + 6y - 9}$.

192. Addition and subtraction of imaginaries. The fundamental operations of addition and subtraction are performed on imaginary and complex numbers as they are performed on rational numbers and ordinary radicals of the same form.

Thus $$2\sqrt{-1}+4\sqrt{-1}=6\sqrt{-1},$$

and $$5\sqrt{-1}-3\sqrt{-1}=2\sqrt{-1}.$$

Also $(3+5\sqrt{-1})+(4-2\sqrt{-1})=7+3\sqrt{-1}.$

Similarly, $$(a+bi)+(c+di)=a+c+(b+d)i.$$

EXERCISES

Simplify:

1. $3\sqrt{-1}+2\sqrt{-1}.$
2. $4\sqrt{-1}+\sqrt{-4}.$
3. $\sqrt{-25}-\sqrt{-16}.$
4. $\sqrt{-9}+\sqrt{-4}.$
5. $\sqrt{-4}+\sqrt{-16}.$
6. $(-8)^{\frac{1}{2}}+(-32)^{\frac{1}{2}}.$
7. $\sqrt{-18}+\sqrt{-8}.$
8. $4\sqrt{-25x^2}-2\sqrt{-36x^2}.$
9. $2+3\sqrt{-1}+6-5\sqrt{-1}.$
10. $7\sqrt{-a^2}-5a+4\sqrt{-a^2}.$
11. $(3a-6ib)+(a+ib).$
12. $4-8i+16-3\sqrt{-9}.$
13. $5+3\sqrt{-49x^2}-6\sqrt{-16x^2}+4.$
14. $18-3(-1)^{\frac{1}{2}}+6(-25)^{\frac{1}{2}}+4.$
15. $5\sqrt{-3}+3\sqrt{-2}-\sqrt{-27}+2\sqrt{-8}.$
16. $(8-5\sqrt{-16})-(7+3\sqrt{-25}).$
17. $4\sqrt{-9a^4}-6a^2\sqrt{-16}+3\sqrt{-6}+5\sqrt{-54}.$
18. $(5x-6iy)-(3x+2iy).$

193. Multiplication of imaginaries. By the definition of square root, the square of $(-n)^{\frac{1}{2}}$ is $-n$.

Therefore $(\sqrt{-1})^2=-1.$

$$(\sqrt{-1})^3=(\sqrt{-1})^2\sqrt{-1}=-\sqrt{-1}.$$

$$(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2=(-1)(-1)=1.$$

To multiply $\sqrt{-2}$ by $\sqrt{-3}$, we first write

$$\sqrt{-2}=\sqrt{2}\cdot\sqrt{-1},$$

and

$$\sqrt{-3}=\sqrt{3}\cdot\sqrt{-1}.$$

Then

$$\sqrt{-2}\cdot\sqrt{-3}=(\sqrt{2}\cdot\sqrt{-1})(\sqrt{3}\cdot\sqrt{-1})$$
$$=\sqrt{6}\cdot\sqrt{-1}\cdot\sqrt{-1}=-\sqrt{6}.$$

Similarly,

$$(2\sqrt{-5})(-3\sqrt{-2})=(2\sqrt{5}\cdot\sqrt{-1})\cdot(-3\sqrt{2}\cdot\sqrt{-1})$$
$$=-6\sqrt{10}(-1)=6\sqrt{10}.$$

In general, if $\sqrt{-a}$ and $\sqrt{-b}$ are two imaginaries whose product is desired, they should first be written in the form $\sqrt{a}\cdot\sqrt{-1}$ and $\sqrt{b}\cdot\sqrt{-1}$ and the multiplication should only then be performed. This method will prevent many errors.

In this connection it must be clearly understood that one rule followed in the multiplication of real radicals (see page 350) does not apply to imaginary numbers.

In the case of ordinary radicals we have

$$\sqrt{2}\cdot\sqrt{3}=\sqrt{2\cdot 3}=\sqrt{6}.$$

But the product of two imaginaries like $\sqrt{-2}\cdot\sqrt{-3}$ does not equal $\sqrt{(-2)(-3)}$, for this equals $\sqrt{6}$. We have seen above that $\sqrt{-2}\cdot\sqrt{-3}=-\sqrt{6}$.

In multiplying two complex numbers, write each expression in the form $a \pm bi$ and proceed as in the following

EXAMPLE

Multiply $4+\sqrt{-5}$ by $2-\sqrt{-6}$.

Solution. $$4+\sqrt{-5}=4+\sqrt{5}\cdot\sqrt{-1}, \qquad (1)$$
$$\underline{2-\sqrt{-6}=2-\sqrt{6}\cdot\sqrt{-1}.} \qquad (2)$$

Multiplying (1) by (2), $8+2\sqrt{5}\cdot\sqrt{-1}-4\sqrt{6}\cdot\sqrt{-1}-\sqrt{30}(-1)$.

Rewriting, $8+2\sqrt{-5}-4\sqrt{-6}+\sqrt{30}$.

EXERCISES

Perform the following indicated multiplications and simplify results :

1. $(-1)^5$.
2. $(-1)^6$.
3. $(-1)^7$.
4. $(-1)^8$.
5. $(\sqrt{-1})^{-4}$.
6. $(\sqrt{-1})^7$.
7. $(\sqrt{-2})^8$.
8. $(\sqrt{-3})^5$.
9. $(2\sqrt{-5})^7$.
10. $\sqrt{-4}\cdot\sqrt{-25}$.
11. $\sqrt{-4}(-\sqrt{-9})$.
12. $\sqrt{-7}(-\sqrt{-6})$.
13. $\sqrt{-16}\cdot\sqrt{-10}$.
14. $2\sqrt{-3}\cdot 3\sqrt{-2}$.
15. $\sqrt{-a}\cdot\sqrt{-b}$.
16. $3\sqrt{-7}(-2\sqrt{-5})$.
17. $\sqrt{m+n}\cdot\sqrt{-m-n}$.
18. $(3+\sqrt{-1})(3-\sqrt{-1})$.
19. $(5+\sqrt{-3})(5-\sqrt{-3})$.
20. $(3-4\sqrt{2}\,i)(3+2\sqrt{2}\,i)$.
21. $(4+\sqrt{-1})(5-\sqrt{-3})$.
22. $(5-3i)(6-5\sqrt{2}\,i)$.
23. $(a+ib)(c+id)$.
24. $(a+ib)^2$.
25. $(a+ib)(a-ib)$.
26. $(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})^2$.
27. $(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})^2$.
28. $(-2+2\sqrt{-3})^3$.
29. $(-2-2\sqrt{-3})^3$.
30. $(x-iy)^3$.
31. $(a+ib)^2-(a-ib)^2$.
32. $(2+2\sqrt{-3})^3-(2-2\sqrt{-3})^3$.
33. $(a+i\sqrt{1-b^2})(a-i\sqrt{1-b^2})$.

34. Determine whether the sum and the product of the numbers $5+6\sqrt{-2}$ and $5-6\sqrt{-2}$ are real.

35. Determine whether the sum and the product of the numbers $2+\sqrt{-3}$ and $2-\sqrt{-3}$ are real.

194. Division of imaginaries. One complex number is the **conjugate** of another if their product and their sum are real. Thus $a+bi$ and $a-bi$ are conjugates. Conjugate complex numbers are used in division of imaginary expressions as conjugate radicals are used in division of real radicals.

In case either the numerator or the denominator of a fraction is imaginary or complex, the division may be performed as in the following

EXAMPLES

1. $\sqrt{-6} \div \sqrt{2}$.

Solution. $$\frac{\sqrt{-6}}{\sqrt{2}} = \frac{\sqrt{-6} \cdot \sqrt{2}}{(\sqrt{2})^2} = \frac{2\sqrt{-3}}{2} = \sqrt{3}\, i.$$

2. $\sqrt{8} \div \sqrt{-2}$.

Solution. $$\frac{\sqrt{8}}{\sqrt{-2}} = \frac{\sqrt{8} \cdot \sqrt{-2}}{(\sqrt{-2})^2} = \frac{4i}{-2} = -2i.$$

3. $\sqrt{-6} \div \sqrt{-2}$.

Solution. $$\frac{\sqrt{-6}}{\sqrt{-2}} = \frac{\sqrt{-6} \cdot \sqrt{-2}}{(\sqrt{-2})^2} = \frac{\sqrt{6} \cdot i \cdot \sqrt{2} \cdot i}{-2} = \sqrt{3}.$$

4. $3 \div (2 + \sqrt{-3})$.

Solution. $$\frac{3}{2+\sqrt{-3}} = \frac{3(2-\sqrt{-3})}{(2+\sqrt{-3})(2-\sqrt{-3})} = \frac{6-3\sqrt{-3}}{4+3}$$
$$= \frac{6-3\sqrt{-3}}{7}.$$

The method of the above examples is stated in the

Rule. *Write the dividend over the divisor in the form of a fraction.*

Then multiply both numerator and denominator of this fraction by the simplest expression which will make the new denominator real and rational.

Reduce the result to its simplest form.

EXERCISES

Perform the indicated operations:

1. $\sqrt{-12} \div \sqrt{2}$.
2. $\sqrt{6} \div \sqrt{-2}$.
3. $2\sqrt{5} \div 4\sqrt{-1}$.
4. $\sqrt{-9} \div \sqrt{-1}$.
5. $1 \div \sqrt{-3}$.
6. $4 \div \sqrt{-5}$.
7. $\sqrt{8} \div \sqrt{-2}$.
8. $(-49)^{\frac{1}{2}} \div (-64)^{\frac{1}{2}}$.
9. $\sqrt{by} \div \sqrt{-y}$.
10. $\sqrt{-m} \div \sqrt{-n}$.
11. $(-6bx)^{\frac{1}{2}} \div (-5x)^{\frac{1}{2}}$.
12. $[(-x^6)^{\frac{1}{2}} \div (-x^2)^{\frac{1}{2}}] \div (-x)^{\frac{1}{2}}$.
13. $3 \div (1 + \sqrt{-1})$.
14. $2 \div (1 - \sqrt{-2})$.
15. $2\sqrt{-5} \div (\sqrt{-1} + 6)$.
16. $2\sqrt{-3} \div (3\sqrt{-2} + 3)$.
17. $(-1 - \sqrt{-3}) \div (-1 + \sqrt{-3})$.
18. $(1 + 2i) \div (3 - 4i)$.
19. $x \div (x + iy)$.
20. $(a + ib) \div (c + id)$.
21. $(3 + 2i)(1 - i) \div (3 - 4i)(1 + i)$.
22. Is $+1 - \sqrt{-3}$ a cube root of -8?
23. Does $x^2 - 6x + 12 = 0$ if $x = 3 \pm \sqrt{-3}$?
24. Does $x = \frac{8}{5}(\sqrt{-10})$, $y = -\frac{8}{5}(\sqrt{-10})$, satisfy the system $x^2 - xy - 12y^2 = 8$, $x^2 + xy - 10y^2 = 20$?

195. Equations with imaginary roots. The student should now be able to check the solution of an equation which has imaginary roots.

EXERCISES

Solve the equations which follow, and check the results:

1. $x^2 + 4x + 8 = 0$.
2. $x^2 - 8x + 24 = 0$.
3. $x^2 + 3x + 9 = 0$.
4. $x^2 - 5x + 16 = 0$.
5. $3x^2 + 2x + 4 = 0$.
6. $x^2 + x + 1 = 0$.
7. $x^2 - x + 1 = 0$.
8. $5x^2 - 6x + 14 = 0$.
9. $6x^2 + 10x + 21 = 0$.
10. $3x^2 + 16x + 21 = 0$.

11. $x^3 = 1$.

Hints. If $x^3 = 1$, $x^3 - 1 = 0$.
Hence $(x-1)(x^2+x+1) = 0$.
Then $x - 1 = 0$,
and $x^2 + x + 1 = 0$.

12. $x^3 + 1 = 0$.
13. $x^3 = 8$.
14. $x^3 = -27$.
15. $x^4 = 1$.
16. $x^4 = 9$.
17. $x^6 = 1$.
18. $x^6 = 64$.
19. $x^3 = 64$.
20. $x^3 = -125$.

21. How many square roots has any real number? cube roots? fourth roots? sixth roots?

22. What do the preceding exercises suggest regarding the number of nth roots which any real number has?

23. $27x^3 - 8 = 0$.
24. $64x^3 + 125 = 0$.
25. $x^4 - 2x^2 - 8 = 0$.
26. $x^3 + x^2 - 2x - 2 = 0$.
27. $x^6 + 7x^3 - 8 = 0$.
28. $3x^4 + 16x^2 + 21 = 0$.
29. $27x^4 - 12x^2 - 64 = 0$.
30. $6x^6 + 21x^3 + 9 = 0$.
31. $25x^4 + 40x^2 + 64 = 0$.
32. $(x^2+4)(x^2+3x+7) = 0$.
33. $(x^2+2x)^2 + 15(x^2+2x) + 54 = 0$.
34. $(x^2+5x)^2 + 9(x^2+5x) - 112 = 0$.

35. Solve $x + y = 4$, $x^2 - 3xy - y^2 = -39$, and check the results.

36. Solve $x + 2y = 4$, $y^2 - x = 0$, and check the results.

37. Solve $x^2 + y^2 = 4$, $x - y = 6$, and check the results.

Note. Long before the time of Gauss, mathematicians had performed the operations of multiplication and division on complex numbers by the same rules that they used for real numbers. As early as 1545 Cardan showed that the product of $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ was the real number 40. However, he was not always equally fortunate in obtaining correct results, for in another place he sets $\frac{1}{4}\left(-\sqrt{-\frac{1}{4}}\right) = \frac{1}{\sqrt{64}} = \frac{1}{8}$.

Even the rather complicated formula for extracting any root of a complex number was discovered in the early part of the eighteenth century. But all these operations were purely formal, and seemed to most mathematicians a mere juggling with symbols until Gauss showed clearly the place and usefulness of such numbers.

196. Factors involving imaginaries. After studying radicals we enlarged our previous notion of a factor and, with certain limitations, employed radicals among the terms of a factor. Now in a similar manner, with like restrictions, we extend our notion of a factor still farther and use imaginary numbers as coefficients or as terms in a factor. For example, x^2+1 may hereafter be regarded as factorable,

for $$x^2+1=x^2-(-1)=(x+\sqrt{-1})(x-\sqrt{-1}).$$

Similarly,

$$4x^2+9=4x^2-(-9)=(2x+3\sqrt{-1})(2x-3\sqrt{-1})$$

and $$x^2+6=x^2-(-6)=(x+\sqrt{-6})(x-\sqrt{-6}).$$

Further, $x^3-1=(x-1)(x^2+x+1)$. Hitherto the trinomial x^2+x+1 has been regarded as prime; but the student can easily prove that x^2+x+1 is equal to the product $(x+\frac{1}{2}+\frac{1}{2}\sqrt{-3})(x+\frac{1}{2}-\frac{1}{2}\sqrt{-3})$. Therefore x^3-1 has three factors, these two and $x-1$.

NOTE ON THE USE OF IMAGINARIES. We have explained the laws of addition, subtraction, multiplication, and division for imaginary (and complex) numbers and have made some use of them. It is largely because imaginaries obey these laws that we call them numbers, for it must be admitted that we cannot count objects with imaginary numbers. Nor can we state by means of them our age, our weight, or the area of the earth's surface. It should be remembered, however, that we can do none of these things with negative numbers. We may have a group of objects — books, for example — whose number is 5; but no group of *objects* exists whose number is -5, or -3, or any negative number whatever. If it is asked, How, then, can negative numbers and imaginary numbers have any practical use? the answer is this: They have a practical use because when they enter into our calculations and we have performed the necessary operations upon them and obtained our final result, that result can frequently be interpreted as a concrete number like those dealt with in ordinary arithmetic. Moreover, if the result cannot be so interpreted, it is, in applied mathematics at least, finally rejected.

In that part of electrical engineering where the theory and measurement of alternating currents of electricity are treated, complex numbers have had extensive use. Their employment in the difficult problems which there arise has given a briefer, a more direct, and a more general treatment than the earlier ones where such numbers are not used.

In theoretical mathematics complex numbers have been of great value in many ways. For example, numerous important theorems about functions are more easily proved under the assumption that the variable is complex. Then, by letting the imaginary part of the complex number become zero, we obtain the proof of the theorem for real values of the variable. Indeed, the student need not go very far beyond this point in his mathematical work to learn that, if e is 2.7182 + (see page 481), $e^{\sqrt{-1}} + e^{-\sqrt{-1}}$ is equal to the real number 1.082 +. At the same time he will learn also how such a form arises, and something of its importance. In a way which we cannot now explain, even so involved an expression as $(a + ib)^{c+id}$ has in higher work a meaning and a use. If the student pursues his mathematical studies far enough, that meaning and use and a multitude of other uses for complex numbers will become familiar to him. But the numbers which we have learned in this book to use, namely fractions, negative numbers, irrational numbers, and complex numbers, complete the number system of ordinary algebra, for it can be proved that from the fundamental operations no other forms of number can arise.

CHAPTER XXIX

THEORY OF QUADRATIC EQUATIONS

197. Formation of equations with given roots. According to section 34, the equation $(x-2)(x-3)=0$ has the roots 2 and 3. In general, the equation $(x-r_1)(x-r_2)=0$ has the roots r_1 and r_2, because either of these numbers, when substituted for x, satisfies the equation. Hence we can always find an equation whose roots are two given numbers r_1 and r_2 by setting the product of the binomials $x-r_1$ and $x-r_2$ equal to zero.

For example, an equation whose roots are 3 and 4 is seen in $(x-3)(x-4)=0$, or $x^2-7x+12=0$.

EXERCISES

Form an equation whose roots are the following:

1. 2, 3.
2. 3, 7.
3. 1, -3.
4. -2, -5.
5. 2, $\frac{2}{5}$.
6. 5, $\frac{3}{7}$.
7. $-\frac{2}{3}$, $\frac{5}{6}$.
8. $-\frac{4}{5}$, $-\frac{7}{8}$.
9. $1+\sqrt{3}$, $1-\sqrt{3}$.
10. $3\pm\sqrt{7}$.
11. $2+\sqrt{-5}$, $2-\sqrt{-5}$.
12. $-7+\sqrt{-5}$, $-7-\sqrt{-5}$.
13. $\frac{1}{2}+\sqrt{\frac{3}{2}}$, $\frac{1}{2}-\sqrt{\frac{3}{2}}$.
14. $\frac{1}{2}+\sqrt{-\frac{3}{2}}$, $\frac{1}{2}-\sqrt{-\frac{3}{2}}$.
15. 2, 3, 4.

HINT. $(x-2)(x-3)(x-4)=0$.

16. 1, 3, 5.
17. 1, -1, 2.
18. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.
19. $\sqrt{-2}$, $-\sqrt{-2}$, 2.
20. $1+\sqrt{2}$, $1-\sqrt{2}$, 3.
21. 1, -1, $\sqrt{-1}$, $-\sqrt{-1}$.
22. $\sqrt{2}-\sqrt{3}$, $\sqrt{2}+\sqrt{3}$, $-\sqrt{2}-\sqrt{3}$, $-\sqrt{2}+\sqrt{3}$.

198. Relations between roots and coefficients. By direct multiplication we obtain from

$$(x-r_1)(x-r_2)=0 \qquad (1)$$

the equation

$$x^2-(r_1+r_2)x+r_1r_2=0. \qquad (2)$$

Since r_1 and r_2 are the roots of (1), it appears from an inspection of (2) that the quadratic equation

$$x^2+bx+c=0$$

has the roots r_1 and r_2, provided $b=-(r_1+r_2)$ and $c=r_1r_2$.

For example, we may form at once the equation whose roots are 4 and 9, as follows:

$$x^2-(4+9)x+4\cdot 9=0, \text{ or } x^2-13x+36=0.$$

Similarly for the cubic equation $x^3+bx^2+cx+d=0$, whose roots are r_1, r_2, and r_3, we have $(x-r_1)(x-r_2)(x-r_3)=0$.

Then
$$b=-(r_1+r_2+r_3),$$
$$c=r_1r_2+r_1r_3+r_2r_3,$$
and
$$d=-r_1r_2r_3.$$

ORAL EXERCISES

Form equations whose roots are the following:

1. 2, 9.

2. 4, 5.

3. −1, 6.

4. −3, −5.

5. −7, 2.

6. $\sqrt{3}, -\sqrt{3}$.

7. $2\sqrt{2}, -2\sqrt{2}$.

8. $3+\sqrt{7}, 3-\sqrt{7}$.

9. $1+\sqrt{2}, 1-\sqrt{2}$.

We will now show the relations which exist between the roots and the coefficients of the general quadratic equation $ax^2+bx+c=0$. By section 188 the roots of $ax^2+bx+c=0$ are

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Adding r_1 and r_2, we have

$$r_1+r_2=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=\frac{-2b}{2a}=-\frac{b}{a}.$$

Therefore $$-(r_1+r_2)=\frac{b}{a}.$$

Multiplying r_1 by r_2, we have

$$r_1r_2=\frac{-b+\sqrt{b^2-4ac}}{2a}\cdot\frac{-b-\sqrt{b^2-4ac}}{2a}$$
$$=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}.$$

Therefore $$r_1r_2=\frac{c}{a}.$$

These results may be expressed verbally as follows:

For the equation $ax^2+bx+c=0$,

I. The sum of the roots with its sign changed is $\frac{b}{a}$.

II. The product of the roots is $\frac{c}{a}$.

These relations frequently afford the simplest means of checking the result of solving a quadratic equation, as illustrated in Exercises 18–31, below.

EXERCISES

Form the equation whose roots are the following:

1. $\frac{3}{2}$, $-\frac{4}{5}$.

Solution. $-\left(\frac{3}{2}-\frac{4}{5}\right)=-\frac{7}{10}=\frac{b}{a}$; $\left(\frac{3}{2}\right)\left(-\frac{4}{5}\right)=-\frac{6}{5}=\frac{c}{a}$.

Hence the required equation is

$$x^2-\frac{7}{10}x-\frac{6}{5}=0, \text{ or } 10x^2-7x-12=0.$$

2. $\frac{4}{3}, \frac{1}{2}$.

3. $\frac{1}{4}, -\frac{2}{3}$.

4. $1\frac{1}{2}, -3\frac{1}{5}$.

5. $4.41, 1.59$.

6. $2 + 3\sqrt{3}, 2 - 3\sqrt{3}$.

7. $4 + \sqrt{-2}, 4 - \sqrt{-2}$.

8. $\frac{3}{2} + \sqrt{7}, \frac{3}{2} - \sqrt{7}$.

9. $\frac{1}{2} + \frac{1}{2}\sqrt{-3}, \frac{1}{2} - \frac{1}{2}\sqrt{-3}$.

10. $1\frac{2}{3} \pm \sqrt{3}$.

11. $\pm\sqrt{-1} + 1$.

12. $a, -\dfrac{1}{a}$.

13. $\dfrac{3a}{2}, \dfrac{5a}{2}$.

14. $1 + a, 1 - a$.

15. $\dfrac{a+b}{a-b}, \dfrac{a-b}{a+b}$.

16. $6, 8, \frac{1}{2}$.

17. $-4, 4, \frac{1}{4}$.

Solve the following equations by the use of the formula and check the result by the use of I and II, above:

18. $x^2 - 5x + 6 = 0$.

19. $x^2 - x - 3 = 0$.

20. $x^2 - 2x - 4 = 0$.

21. $x^2 - 9x - 10 = 0$.

22. $x^2 + 2x + 1 = 0$.

23. $x^2 + 8x + 16 = 0$.

24. $x^2 + 5x + 5 = 0$.

25. $2x^2 + 3x - 6 = 0$.

26. $3x^2 + 3x - 5 = 0$.

27. $5x^2 - 6x + 10 = 0$.

28. $x^2 + x + 1 = 0$.

29. $\dfrac{x^2}{3} + \dfrac{x}{4} + \dfrac{1}{5} = 0$.

30. $\dfrac{x^2}{2} + 4x - 7 = 0$.

31. $6x^2 - 4x - 3 = 0$.

Find the value of the literal coefficient in the following:

32. $x^2 + 2x - c = 0$, if one root is 2.

33. $x^2 - x - c = 0$, if one root is 6.

34. $x^2 - cx - 70 = 0$, if one root is 7.

35. $x^2 + 2bx + 20 = 0$, if one root is -4.

36. $x^2 - 8x + c = 0$, if one root is twice the other.

37. $x^2 + 7x + c = 0$, if one root exceeds the other by 2.

38. $x^2 + 11x + b = 0$, if the difference between the roots is 10.

199. Character of the roots of a quadratic equation. It is often desirable to determine whether the roots of a given quadratic equation are real or imaginary, rational or irrational, equal or unequal, without solving the equation. This can be accomplished by use of the formulas for the roots of the quadratic $ax^2 + bx + c = 0$:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \tag{1}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

These expressions are seen to differ from each other only in the sign preceding the radical. The expression

$$b^2 - 4ac,$$

which appears under the radical sign, is called the **discriminant** of the quadratic. The only way in which r_1 or r_2 can be a complex number is for the discriminant to be negative. If all of the coefficients a, b, and c are rational, r_1 or r_2 can be rational only when $\sqrt{b^2 - 4ac}$ is rational; that is, when the discriminant is a perfect square.

Hence if a, b, and c are rational, an inspection of (1) and (2) shows that the following statements are true:

I. If $b^2 - 4ac$ is positive and not a perfect square, the roots are real, unequal, and irrational.

For example, in $x^2 - 8x + 2 = 0$ the discriminant $b^2 - 4ac$ equals $(-8)^2 - 4 \cdot 1 \cdot 2 = 56$, which is not a perfect square. The roots of the equation are the real, unequal, and irrational numbers $4 \pm \sqrt{14}$.

II. If $b^2 - 4ac$ is positive and a perfect square, the roots are real, unequal, and rational.

For example, in the equation $2x^2 - 3x - 9 = 0$ the discriminant $b^2 - 4ac$ equals $(-3)^2 - 4 \cdot 2 \cdot (-9) = 81$, which is a perfect square. The roots are the real, unequal, rational numbers $-\frac{3}{2}$ and $+3$.

III. If $b^2 - 4ac$ is zero, the roots are equal.

For example, in the equation $4x^2 - 12x + 9 = 0$, the discriminant $b^2 - 4ac$ equals $(-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$. The only number which satisfies this equation is $\frac{3}{2}$, so in one sense the equation has only one root. But since the left member has two identical factors each of which affords the same root of the equation, it is customary to say that the equation has equal roots.

IV. If $b^2 - 4ac$ is negative, the roots are imaginary.

For example, in the equation $2x^2 - 5x + 4 = 0$ the discriminant $b^2 - 4ac$ equals $(-5)^2 - 4 \cdot 2 \cdot 4 = 25 - 32 = -7$. The roots of the equation are the conjugate imaginaries $\frac{+5 + \sqrt{-7}}{4}$ and $\frac{+5 - \sqrt{-7}}{4}$.

ORAL EXERCISES

Determine the character of the roots of the following equations without solving:

1. $x^2 + 5x + 6 = 0$.
2. $3x^2 - 4x + 1 = 0$.
3. $2x^2 - 6x - 3 = 0$.
4. $2x^2 - 3x - 2 = 0$.
5. $5x^2 - 5x + 4 = 0$.
6. $x^2 + x + 1 = 0$.
7. $x^2 - 8x + 16 = 0$.
8. $2x^2 + 3x + 5 = 0$.
9. $4x^2 - 4x + 1 = 0$.
10. $3x^2 - 2x - 2 = 0$.

EXERCISES

Determine the value of k which will make the roots of the following equations equal:

1. $x^2 - kx + 16 = 0$.

Solution. $a = 1$, $b = -k$, $c = 16$.

Hence $b^2 - 4ac = k^2 - 64$.

In order for the roots to be equal, $b^2 - 4ac$ must equal zero.

Therefore $k^2 - 64 = 0$.

Whence $k = \pm 8$.

Check. Substituting 8 for k in the original equation,

$$x^2 - 8x + 16 = 0.$$

Whence $x = 4$, only.

Similarly, substituting -8 for k,

$$x^2 + 8x + 16 = 0.$$

Whence $x = -4$, only.

2. $x^2 + kx + 16 = 0.$
3. $x^2 - 10x + k = 0.$
4. $2x^2 + 8x + k = 0.$
5. $x^2 - 3kx + 36 = 0.$
6. $3x^2 + 4kx + 12 = 0.$
7. $9x^2 + 30x + k + 9 = 0.$
8. $4kx^2 - 60x + 25 = 0.$
9. $9k^2x^2 - 84x + 49 = 0.$
10. $49x^2 - (k+3)x + 4 = 0.$
11. $(k^2+5)x^2 - 30x + 25 = 0.$

Determine the values of a for which the following systems will have two sets of equal roots:

12. $y^2 = 2x,\ y = x + a.$
13. $x^2 + y^2 = a^2,\ y = x + 1.$
14. $xy = a,\ x + y = 1.$

200. Number of roots of a quadratic. In section 187 we found that every quadratic equation has two roots.

Up to the present we have assumed that a quadratic equation can have no other root than the ones found by the method of completing the square. This fact can be proved as follows:

Proof. If we write the equation $ax^2 + bx + c = 0$ in the form $x^2 + \frac{b}{a} \cdot x + \frac{c}{a} = 0$ and substitute therein from (I) and (II) on page 398, we get $x^2 - (r_1 + r_2)x + r_1r_2 = 0$. This can be factored and written as $(x - r_1)(x - r_2) = 0$. Now if any value of x different from r_1 and r_2, say r, be a root of this equation, such a value when substituted for x must satisfy the equation $(x - r_1)(x - r_2) = 0$.

Hence $(r - r_1)(r - r_2)$ must equal zero. By assumption, however, r is different from r_1 and r_2. Consequently neither the factor $r - r_1$

nor $r - r_2$ can equal zero, and therefore their product cannot equal zero. This proves that no additional value r can satisfy the equation $x^2 - (r_1 + r_2)x + r_1r_2 = 0$. As this equation is but another form of $ax^2 + bx + c = 0$, the latter has only two roots.

201. Factors of quadratic expressions. Let r_1 and r_2 be the roots of $ax^2 + bx + c = 0$; then

$$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (r_1 + r_2)x + r_1r_2 = (x - r_1)(x - r_2),$$

or
$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

Therefore the three factors a, $x - r_1$, and $x - r_2$ of any quadratic expression can be found if we first set the expression equal to zero (see section 69) and solve the equation thus formed. Obviously the character of the roots so obtained will determine the character of the factors. Hence by the use of the discriminant $b^2 - 4ac$ we can decide whether the factors of a quadratic expression are real or imaginary, rational or irrational, without factoring it.

EXERCISES

Determine which of the following expressions have rational factors:

1. $x^2 - 3x - 40$.
2. $2x^2 + 5x - 7$.
3. $7x^2 - 9x + 18$.
4. $24x^2 - x - 10$.
5. $72x^2 - 17x + 1$.
6. $5x^2 + 3x - 20$.
7. $3x^2 - 9x + 28$.
8. $33h^2 - 233h - 6$.
9. $x^2 - 2ax + (a^2 - b^2)$.
10. $abx^2 - (b^2 + a^2)x + ab$.

Separate into rational, irrational, or imaginary factors:

11. $2x^2 + 5x - 8$.

Solution. Let $2x^2 + 5x - 8 = 0$.

Solving by formula, $x = \dfrac{-5 \pm \sqrt{25 - (-64)}}{4} = \dfrac{-5 \pm \sqrt{89}}{4}$.

Then $r_1 = \dfrac{-5+\sqrt{89}}{4}$ and $r_2 = \dfrac{-5-\sqrt{89}}{4}$.

Therefore $2x^2 + 5x - 8 = 2\left[x - \dfrac{-5+\sqrt{89}}{4}\right]\left[x - \dfrac{-5-\sqrt{89}}{4}\right]$

$$= \tfrac{1}{8}(4x + 5 - \sqrt{89})(4x + 5 + \sqrt{89}).$$

12. $x^2 - 7x - 7$.

13. $x^2 - 4x - 1$.

14. $x^2 + 2x + 2$.

15. $x^2 + 4x - 9$.

16. $4x^2 - 12x - 9$.

17. $25x^2 + 20x + 4$.

18. $6x^2 + 14x - 40$.

19. $10 - 9x - 9x^2$.

20. $10x^2 + 12 - 26x$.

21. $x^2 + 7x + 8$.

22. $x^2 + x + 1$.

23. $x^2 + 1$.

24. $x^2 + 9$.

25. $x^2 - 2ax + a^2 - b$.

26. $x^2 + 6ax + 9a^2 - 4b$.

27. $4x^2 + 4ax + a^2 - 4c$.

28. $x^2 - 4ax + 4a^2 + c$.

29. $ax^2 + bx + c$.

30. $x^2 - xy + 5x - 2y + 6$.

Solution. Let $x^2 - xy + 5x - 2y + 6 = 0$.

Then $x^2 + (5 - y)x - 2y + 6 = 0$.

Solving for x in terms of y by the formula,

$$x = \frac{-(5-y) \pm \sqrt{(5-y)^2 - 4(-2y+6)}}{2}$$

$$= \frac{-5 + y \pm \sqrt{y^2 - 2y + 1}}{2}.$$

Whence $x = -2$ and $y - 3$.

Therefore $x^2 - xy + 5x - 2y + 6 = (x + 2)(x - y + 3)$.

31. $3x^2 - 6xy + 14x - 4y + 8$.

32. $x^2 - xy - 2y^2 + 3x - 6y$.

33. $x^2 - 4xy - y + 3y^2 - 2 - x$.

CHAPTER XXX

GRAPHS OF QUADRATIC EQUATIONS IN TWO VARIABLES

202. Graph of a quadratic equation in two variables. Before solving graphically a quadratic system, the method of graphing *one* quadratic equation in *two* variables must be clearly understood.

EXAMPLES

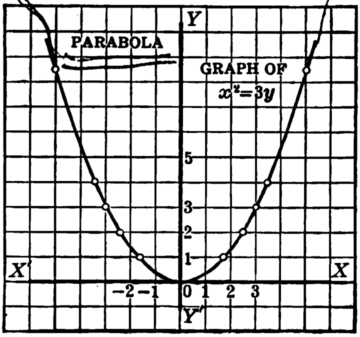
$x^2/3$

1. Construct the graph of $x^2 = 3y$.

Solution. Solving the equation for y, $y = \frac{x^2}{3}$.

We now assign values to x and then compute the approximate corresponding values of y. Tabulating the results gives:

If $x =$	6	4	3	2	1	0	-1	-2	-4	-6
then $y =$	12	$\frac{16}{3}$	3	$\frac{4}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{16}{3}$	12



Using an x-axis and a y-axis as in graphing linear equations, plotting the points corresponding to the real numbers in the table, and drawing the curve determined by these points, we obtain the graph of the adjacent figure. The curve is called **a parabola.**

The graph of any equation of the form $x^2 = ay$ is a parabola.

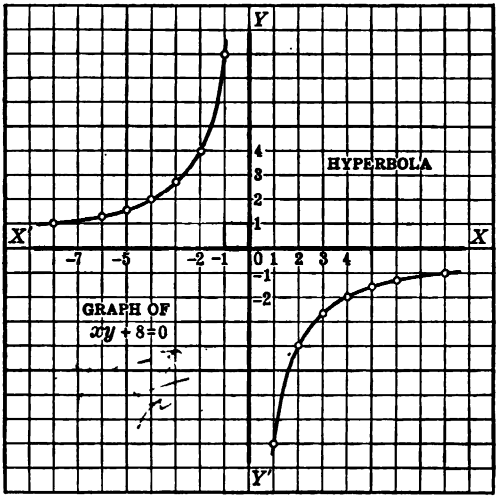
2. Graph the equation $xy + 8 = 0$.

Solution. Solving for y, $y = -\frac{8}{x}$.

Assigning values to x as indicated in the following table, we then compute the corresponding values of y:

If $x =$	-6	-5	-4	-3	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4	5	6	8
then $y =$	$\frac{4}{3}$	$\frac{8}{5}$	2	$\frac{8}{3}$	4	8	16	-16	-8	-4	$-\frac{8}{3}$	-2	$-\frac{8}{5}$	$-\frac{4}{3}$	-1

Proceeding as before with the numbers in the table, we obtain the two-branched curve of the figure below, which does not touch either axis. The curve is called a **hyperbola.**



The graph of any equation of the form $xy = K$ is a *hyperbola.* The curve for $xy = K$ ($K =$ any constant) is always in the same general position; that is, if K is positive, one branch of the curve lies in the first quadrant and the other branch in the third. If K is negative, one branch lies in the second quadrant and the other in the fourth.

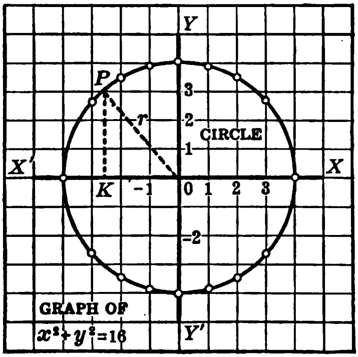
3. Graph the equation $x^2 + y^2 = 16$.

Solution. Solving for y, $y = \pm\sqrt{16 - x^2}$.

Assigning values to x as indicated in the following table, we obtain from page 502 the corresponding values of y:

If $x =$	-5	-4	-3	-2	-1	0	1	2	3	4	5
then $y =$	$\pm 3\sqrt{-1}$	0	± 2.64	± 3.46	± 3.87	± 4	± 3.87	± 3.46	± 2.64	0	$\pm 3\sqrt{-1}$

For values of x numerically greater than 4 it appears that y is imaginary. The points corresponding to the pairs of real numbers in the table lie on the circle in the accompanying figure. The center of the circle is at the origin, and the radius is 4.



The graph of any equation of the form $x^2 + y^2 = r^2$ is a circle whose radius is r. This can be proved from the right triangle PKO. If P represents *any* point on the circle, OK equals the x-distance of P, KP equals the y-distance, and OP equals the radius. Now $\overline{OK}^2 + \overline{KP}^2 = \overline{OP}^2$; that is, $x^2 + y^2 = r^2$. It follows, then, that the graphs of $x^2 + y^2 = 9$ and $x^2 + y^2 = 8$ are circles whose centers are at the origin and whose radii are 3 and $\sqrt{8}$ respectively. Hereafter, when it is required to graph an equation of the form $x^2 + y^2 = r^2$, the student may use compasses and, with the origin as the center and with the proper radius (the square root of the constant term), describe the circle at once.

In all the graphical work which follows, the student will save time by obtaining from the table on page 502 the square roots or cube roots which he may need.

4. Graph the equation $16x^2 + 9y^2 = 144$.

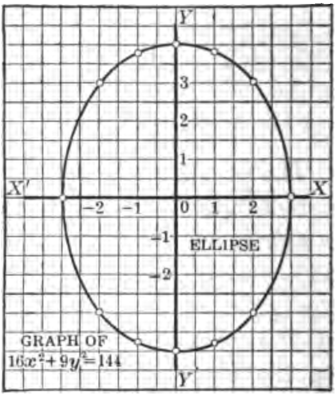
Solution. Solving, $y = \pm \frac{4}{3}\sqrt{9 - x^2}$. Proceed as in Example 3:

$y^2 = \frac{144 - 16x^2}{9}$

If $x =$	-4	-3	-2	-1	0	$+1$	$+2$	$+3$	$+4$
then $y =$	$\pm\frac{4}{3}\sqrt{-7}$	0	± 2.98	± 3.77	± 4	± 3.77	± 2.98	0	$\pm\frac{4}{3}\sqrt{-7}$

$y = \pm\sqrt{144 - 16x^2}$

For any value of x numerically greater than 3, y is imaginary. The points corresponding to the real numbers in the table lie on the graph of the adjacent figure. The curve is called an **ellipse.**



The graph of any equation of the form of $ax^2 + by^2 = c$ in which a and b are unequal and of the same sign as c is an *ellipse.*

NOTE. These three curves — the ellipse, the hyperbola, and the parabola — were first studied by the Greeks, who proved that they are the sections which one obtains by cutting a cone by a plane. Not for hundreds of years afterwards did anyone imagine that these curves actually appear in nature, for the Greeks regarded them merely as geometrical figures, and not at all as curves that have anything to do with our everyday life. One of the most important discoveries of astronomy was made by Kepler (1571–1630), who showed that the earth revolves around the sun in an ellipse, and stated the laws which govern the motion. Those comets that return to our field of vision periodically also have elliptic orbits, while those that appear once, never to be seen again, describe parabolic or hyperbolic paths.

The path of a ball thrown through the air in any direction, except vertically upward or downward, is a parabola. The approximate parabola which a projectile actually describes depends on the elevation of the gun (the angle with the horizontal), the quality of the powder, the amount of the charge, the direction of the wind, and various other conditions. This makes gunnery a complex subject.

411

EXERCISES

Construct the graphs of the following equations and state the name of each curve obtained:

1. $x^2 = 4y.$ Parab ~~**3.**~~ $x^2 + y^2 = 49.$ **5.** $x^2 - y^2 = 16.$

2. $y^2 + 2x = 0.$ Parab ~~**4.**~~ $x^2 + y^2 = 18.$ **6.** $xy = 12.$

~~**7.**~~ $xy = -6.$ **9.** $16x^2 - 9y^2 = 144.$

8. $9x^2 + 4y^2 = 36.$ **10.** $25x^2 + 9y^2 = 225.$

203. Graphical solution of a quadratic system in two variables. That we may solve a system of two quadratic equations by a method similar to that employed in section 44 for linear equations appears from the

4

$\sqrt{144 - 16x^2}$

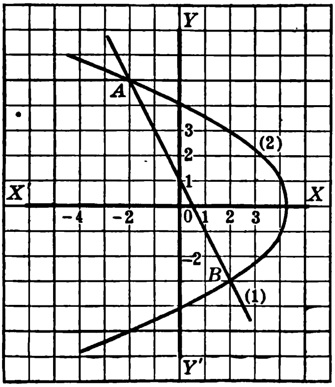
16 9

EXAMPLES

1. Solve graphically $\begin{cases} 2x + y = 1, & (1) \\ y^2 + 4x = 17. & (2) \end{cases}$

Solution. Constructing the graphs of (1) and (2), we obtain the straight line and the parabola shown in the adjacent figure. There are two sets of roots corresponding to two points of intersection, which are

$$A\begin{cases} x = -2, \\ y = 5. \end{cases} \qquad B\begin{cases} x = 2, \\ y = -3. \end{cases}$$

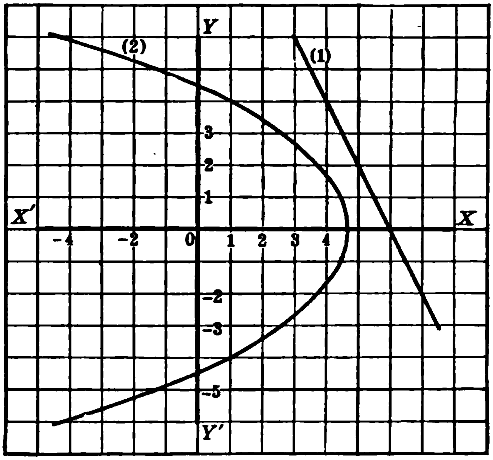


NOTE. If the straight line in the adjacent figure were moved to the right in such a way that it always remained parallel to its present position, the points A and B would approach each other and finally coincide. The line would then be tangent to the parabola at the point $x = 4$, $y = 1$.

Were the straight line moved still farther, it would neither touch nor intersect the parabola and there would be no graphical solution (see page 410).

2. Solve graphically $\begin{cases} 2x + y = 12, & (1) \\ y^2 + 4x = 19. & (2) \end{cases}$

Solution. The graphs of (1) and (2) are the straight line and the parabola of the adjacent figure. These curves have no real points of intersection. There are, however, two pairs of imaginary roots. Solving (1) and (2) by substitution, $x = \frac{11}{2} + \sqrt{-1}$ or $\frac{11}{2} - \sqrt{-1}$, and $y = 1 - 2\sqrt{-1}$ or $1 + 2\sqrt{-1}$.



The essential point to be emphasized here is that real roots of a simultaneous system correspond to real intersections, and imaginary roots correspond to no intersections of real graphs.

3. Solve graphically $\begin{cases} x^2 - y^2 = 4, & (1) \\ y^2 - x - 6 = 0. & (2) \end{cases}$

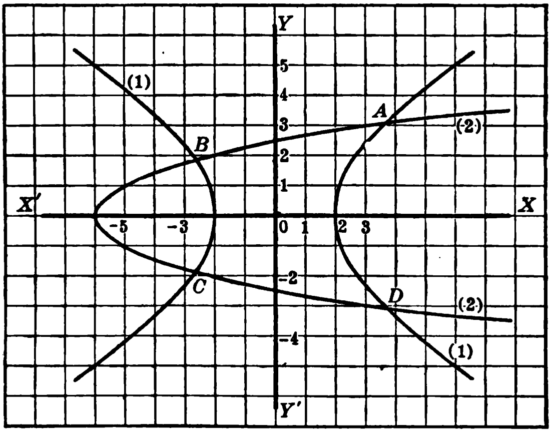
Solution. Constructing the graphs of (1) and (2), we obtain the hyperbola and the parabola of the figure on the opposite page. There are four sets of roots corresponding to the four points of intersection, which are approximately

$$A\begin{cases} x = 3.7, \\ y = 3.1. \end{cases} \quad B\begin{cases} x = -2.7, \\ y = 1.8. \end{cases} \quad C\begin{cases} x = -2.7, \\ y = -1.8. \end{cases} \quad D\begin{cases} x = 3.7, \\ y = -3.1. \end{cases}$$

If the two curves had been so chosen as to intersect only twice, their equations would have had only two sets of real roots.

Examples 1, 2, and 3 partially illustrate the truth of the following statement:

If in a system of two equations in two variables one equation is of the *m*th degree and one of the *n*th, there are *usually mn* sets of roots (real or imaginary) *and never more than mn such sets.*



EXERCISES

If possible, solve graphically each of the following systems:

[illegible] $y^2 = 4x,$ $3x + y = 5.$

[illegible] $x^2 + y^2 = 25,$ $[illegible] - 2x = 10.$

[illegible] $xy = 6,$ $3x + y = 9.$

[illegible] $x^2 + y^2 = 25,$ $x^2 + y^2 = 16.$

5. $x^2 + y^2 = 9,$ $x + y = 10.$

6. $x^2 + y^2 = 36,$ [illegible] $- y^2 = 25.$

7. $x^2 + y^2 = 9,$ $x^2 - y^2 = 16.$

[illegible] $xy = 12,$ $2x + y = 10.$

[illegible] $x^2 = 4y,$ [illegible] $y^2 = 9.$

[illegible] $4y = 17,$ [illegible] $y = 9.$

1[illegible]. $x^2 + 4y = 17,$ $x + 2y = 12.$

12. $x^2 + y = 5,$ $y^2 + x = 3.$

13. $x - y\sqrt{8} = 0,$ $x^2 = y^3 - 9y.$

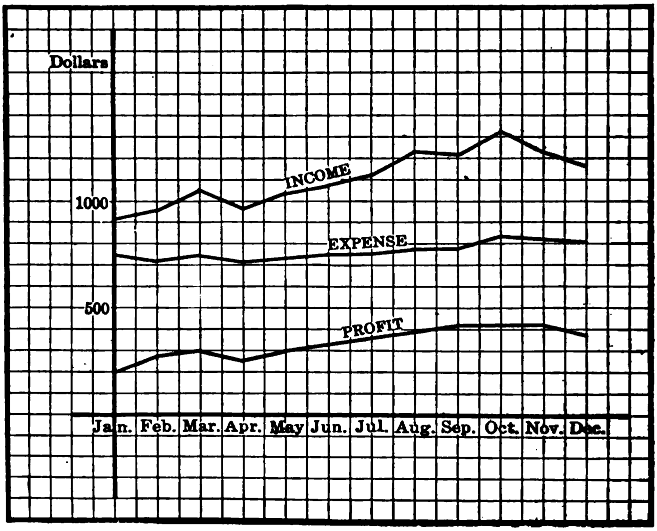
204. Graphical presentation of numerical data. A great variety of statistics can be presented graphically in a very striking manner. Business and commercial houses have during the past few years used the method extensively not only to present facts but also to aid in interpreting them and in indicating their tendencies.

The following exercises illustrate some of the possibilities in the graphical presentation of numerical data.

EXAMPLE

For the year 1912 the total income and expenses per mile of line of all the railroads in the United States having a yearly revenue of one million dollars or more was as follows:

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Income in dollars .	930	970	1050	980	1040	1080	1120	1220	1210	1320	1220	1170
Expenses in dollars	730	710	750	720	740	750	760	780	780	830	810	800



70 175

The foregoing graph represents the given data on the same axes.

The lowest curve on the diagram shows the profits of the railroads. It was obtained by plotting the differences of the numbers in the table.

EXERCISES

1. The average incomes of 155 members of a certain college class for the first ten years after their graduation is given in the following table:

Years after graduation	1	2	3	4	5	6	7	8	9	10
Income in dollars . . .	706	902	1199	1651	2039	2408	2382	2709	3222	3804

Graph the data. What tendencies do you note?

2. The number of inches of rainfall during the month of July and the number of bushels of corn yielded per acre for a term of years in a certain locality is given in the following table:

Year	'89	'90	'91	'92	'93	'94	'95	'96	'97	'98	'99	'00	'01	'02
Rainfall	5.4	2.6	5.1	3.7	3.4	1.9	4.8	5.4	3.7	4.3	4.6	4.7	1.2	6.0
Corn yield	32	23	27	27	24	18	30	38	25	26	28	30	19	32

Plot these data on the same axes. How do you account for the similarity of the curves?

3. The numbers of hundreds of telephone calls in certain business and residential sections of New York City are given in the following table for various hours of the day. Plot both sets of data on the same axes and explain the reason for differences in the shapes of the graphs.

Time of day	7	8	9	10	11	Noon	1	2	3	4	5	6	7	8	9	10	11	12
Business district .	1	2	4	55	108	101	78	75	93	87	67	35	8	2	1.5	1	1	1
Residence district	.25	8	35	73	76	65	52	48	47	40	38	38	36	32	25	15	4	3

4. Measurements of the breadth of the heads of a thousand students in a certain school were as follows:

Head breadth in inches	5.5	5.6	5.7	5.8	5.9	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8
Number of students . .	3	12	43	80	131	236	185	142	99	37	15	12	3	2

Construct a graph of the above data.

5. The chest measurements of 10,000 soldiers were tabulated as follows:

Chest measurement in inches	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
Number of soldiers	5	31	141	322	732	1305	1867	1882	1628	1148	645	160	87	38	7	2

Construct a graph of the above data.

It may appear accidental that the foregoing measurements should group themselves with any regularity. But if the number of measurements of this type is large and each is made with care, they obey a law called the law of probability. In fact the graph of the data in Exercises 4 and 5 is a close approximation to what is called the probability curve.

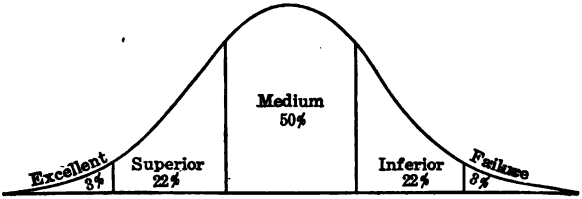
The equation of this curve is $y = e^{-x^2}$ when $e = 2.7$ approximately.

6. Construct the graph of the equation $y = e^{-x^2}$ between the values $x = -2$ and $x = 2$.

HINT. Let $x = -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0$, etc.

NOTE. It is well established that physical characteristics, such as those illustrated by the graphs of Exercises 4 and 5, obey the law of probability. If the graph of Exercise 4 is carefully considered, it may raise the question, Do mental characteristics also obey the law? An interesting aspect of this is given by the fact that an increasing number of high schools and colleges assume that such is the case, and grade, or mark, their students by a system based on the law of probability. Such a system assumes that if one hundred or more students in any subject are examined, the number of students and their degree of mastery of the subject arrange themselves according to the probability curve shown below. Obviously between very many students the differences in grades attained will be small, between many others they will be moderate, and between only a few will they be great.

The percentages found in the figure indicate the part of the total area under the curve lying below each section. Here we see that 3% of the students receive the Excellent mark; 22%, the Superior mark; 50%, the Medium mark; 22%, the Inferior mark; and 3%, the Failure mark.



In the statistical study of problems which have their origin quite remote from each other, this curve frequently occurs, and occupies a central position in the mathematical theory of statistics.

CHAPTER XXXI

SYSTEMS SOLVABLE BY QUADRATICS

205. Introduction. The general equation of the second degree in two variables is $ax^2 + by^2 + cxy + dx + ey + f = 0$. To solve a pair of such equations requires the solution of an equation of the fourth degree. Even the solution of $x^2 + y = 5$ and $y^2 + x = 3$ requires the solution of such an equation. In fact only a limited number of systems of the second degree in two variables is solvable by quadratics. By the graphical methods of Chapter XXX the student can solve graphically for real roots any system of quadratic equations, provided the terms have numerical coefficients. The algebraic solution of such systems will in many cases be possible for him only after further study of algebra.

206. Linear and quadratic systems. Every system of equations in two variables in which one equation is *linear* and the other *quadratic* can be solved by the method of *substitution.*

EXAMPLE

Solve the system
$$\begin{cases} x^2 - 2xy = 7, & (1) \\ x + 2y = 13. & (2) \end{cases}$$

Solution. Solving (2) for y in terms of x,

$$y = \frac{13 - x}{2}. \qquad (3)$$

Substituting $\frac{13 - x}{2}$ for y in (1),

$$x^2 - 2x\left(\frac{13 - x}{2}\right) = 7. \qquad (4)$$

From (4), $$2x^2 - 13x - 7 = 0. \qquad (5)$$

Solving (5), $x = 7 \text{ or } -\frac{1}{2}.$

Substituting 7 for x in (3),

$$y = \frac{13 - 7}{2}$$
$$= 3.$$

Substituting $-\frac{1}{2}$ for x in (3),

$$y = \frac{13 + \frac{1}{2}}{2}$$
$$= 6\frac{3}{4}.$$

The two sets of roots are $x = 7$, $y = 3$, and $x = -\frac{1}{2}$, $y = 6\frac{3}{4}$.

Check. Substituting 7 for x and 3 for y in (1) and (2),

$$49 - 42 = 7,$$
$$7 + 6 = 13.$$

Substituting $-\frac{1}{2}$ for x and $6\frac{3}{4}$ for y in (1) and (2),

$$\frac{1}{4} + \frac{27}{4} = 7,$$
$$-\frac{1}{2} + \frac{27}{2} = 13.$$

EXERCISES

Solve the following systems, pair results, and check each set of roots:

1. $x + y = 5,$
   $x^2 + y^2 = 13.$
2. $x^2 - y^2 = 8,$
   $2x + y = 7.$
3. $x^2 + xy = 15,$
   $x + y = 3.$
4. $x^2 + 2xy = 21,$
   $x + 2y = 3.$
5. $x^2 + 2y = 17,$
   $2x - y = 2.$
6. $y^2 - 10x = 6,$
   $x + 3y = 13.$
7. $y^2 - 2x^2 = 7,$
   $2y + 3x = 1.$
8. $2x^2 - xy = 70,$
   $x + 4y = 23.$
9. $s^2 + 2st = 108,$
   $4s - 3t = 6.$
10. $s^2 - 3st = 22,$
    $4s + 2t = 2.$
11. $s^2 + t^2 = 169,$
    $2s + t = 22.$
12. $s^2 + 2t^2 = 27,$
    $4t - 2s = 18.$

13. $s^2 + 3ts + t^2 = 44,$
$2s - t = 0.$

14. $s^2 + ts + t^2 = 12,$
$s + t = 2.$

15. $2s^2 - ts + t^2 = 16,$
$2s - t = 5.$

16. $xy + 10 = 0,$
$4x + y = 6.$

17. $4x + y = 28,$
$2x^2 + 3xy = 98.$

18. $3x_1 + 4x_2 = 5,$
$2x_1x_2 - 6x_1 = -3.$

19. $\frac{x}{y} + \frac{y}{x} = 2,$
$3x + 2y = 5.$

20. $\frac{1}{x} - \frac{1}{y} = \frac{1}{4},$
$3x - y = 2.$

21. $\frac{x}{4} + \frac{y}{5} = 6,$
$\frac{10}{y} - \frac{10}{x} = \frac{3}{2}.$

22. $\frac{1}{x} + \frac{y}{2} = 3,$
$\frac{1}{y} + \frac{x}{2} = \frac{5}{4}.$

23. $\frac{s}{3} + \frac{t}{4} = 5,$
$\frac{6}{s} - \frac{4}{t} = \frac{2}{3}.$

24. $\frac{t+1}{s+1} = \frac{3}{2},$
$\frac{t+s^2}{s+t^2} = \frac{1}{2}.$

25. $\frac{x}{y} - \frac{y}{x} = \frac{5}{6},$
$3y + 2x = 2.$

26. $s^2 + t^2 + 2t = 40,$
$s + t + 2 = 0.$

27. $2x^2 + xy = 8,$
$y - x = \sqrt{2}.$

28. $x^2 + y^2 + 4x + 6y = 40,$
$x - 10 = y.$

29. $y + x\sqrt{15} = 0,$
$y^2 + x^3 = 16x.$

207. Homogeneous equations. If both the equations of a system are *quadratic*, an attempt to solve it by *substitution* usually gives an equation of the fourth degree. In most cases such an equation could not be solved by factoring, and at the present time its solution by any other method is beyond the student. With certain types of systems, however, which occur more or less frequently, we can employ special devices and avoid the solution of any equation of higher degree than a quadratic. Among these systems are the so-called "homogeneous" systems.

An equation is **homogeneous** if, on being written so that one member is zero, the terms in the other member are of the same *degree* with respect to the variables.

Thus $x^2 + y^2 = xy$ and $x^2 - 3xy + y^2 = 0$ are homogeneous equations of the second degree. $2x^3 + y^3 = x^2y - 3xy^2$ is a homogeneous equation of the third degree.

The system $\left\{\begin{array}{l} xy + y^2 = x^2, \\ x^2 + 2y^2 - 3xy = 0 \end{array}\right\}$ is a homogeneous system.

Systems like $\left\{\begin{array}{l} y^2 + xy + x^2 = 7, \\ 2x^2 - y^2 + 4xy = 6 \end{array}\right\}$ are often called homogeneous systems, but strictly speaking they are not. As will be seen, the method of solving such systems is about the same as the method of solving a homogeneous system. Hence they are classed with homogeneous systems.

208. Systems having both equations quadratic. Occasionally when both equations are quadratic the terms which occur in the two are so related that the elimination of terms involving both variables or of the constant terms can be performed. The following system is of such type:

EXAMPLES

1. Solve the system
$$\left\{\begin{array}{ll} x^2 - xy = 2, & (1) \\ 3x^2 + 4xy = 20. & (2) \end{array}\right.$$

Solution. First eliminate xy by addition (§ 47),

(1) · 4, $\quad 4x^2 - 4xy = 8. \quad$ (3)

(2) + (3), $\quad 7x^2 = 28.$

Whence $\quad x = \pm 2.$

Substituting $+2$ for x in (1), $4-2y=2$, whence $y=1$.

Substituting -2 for x in (1), $4+2y=2$, whence $y=-1$.

Therefore $x=2,\ -2,$

and $y=1,\ -1.$

These values may be checked as usual.

The method of solving a system in which every term in each equation except the constant terms is of the second degree is as follows:

2. Solve $\begin{cases} xy+3y^2=6. & (1) \\ x^2+y^2=10. & (2) \end{cases}$

Solution. First we combine the two equations to obtain a homogeneous equation:

(1) · 5, $5xy+15y^2=30.$ (3)

(2) · 3, $3x^2+3y^2=30.$ (4)

(3) – (4), $-3x^2+5xy+12y^2=0.$ (5)

Solving (5) for x in terms of y, $x=3y,$ (6)

and $x=-\frac{4y}{3}.$ (7)

Substituting $3y$ for x in (2), $9y^2+y^2=10.$ (8)

From (8), $y=\pm 1.$ (9)

Substituting from (9) in (6), $x=\pm 3.$

Substituting $-\frac{4y}{3}$ for x in (2), $\frac{16y^2}{9}+y^2=10.$ (10)

From (10), $y=\pm\frac{3}{5}\sqrt{10}.$ (11)

Substituting from (11) in (7), $x=\mp\frac{4}{5}\sqrt{10}.$

When $x=$	3	-3	$+\frac{4}{5}\sqrt{10}$	$-\frac{4}{5}\sqrt{10}$
then $y=$	1	-1	$-\frac{3}{5}\sqrt{10}$	$+\frac{3}{5}\sqrt{10}$

Each pair of values can be checked as usual.

A quadratic system in which one equation is homogeneous is easier to solve than the system of Example 2, as can be seen from what follows:

3. Solve the system $\begin{cases} 3x^2 + 2y^2 = 7xy, & (1) \\ x^2 + y^2 - 5x = 3. & (2) \end{cases}$

HINTS. Solving (1) for x, $x = 2y,$ (3)

and $x = \frac{y}{3}.$ (4)

We may now substitute from (3) in (2) and from (4) in (2), solve the resulting equations, and then complete the solution as in Example 2.

EXERCISES

Solve, pair results, and check each set of real roots:

1. $x^2 + y^2 = 20,$
$3y^2 + x^2 = 28.$

2. $3x^2 + 2xy = 7,$
$2x^2 - 3xy = -4.$

3. $3y^2 - xy = 2,$
$2y^2 + 3xy = 38.$

4. $y^2 - xy = 2x^2,$
$2x^2 + xy = 16.$

5. $x^2 - 3y + 3 = 0,$
$y^2 + x^2 = 25.$

6. Solve Example 3 completely.

7. $x^2 + 2xy - y^2 = 32,$
$2x^2 - 3xy + y^2 = 0.$

8. $2x^2 - xy + 2y^2 = 12,$
$2x^2 + xy + 2y^2 = 8.$

9. $2x^2 + y^2 = 3xy,$
$x^2 + 3xy = 16.$

10. $s^2 - 3st = 4,$
$t^2 + s^2 = 4.$

11. $s^2 + 2st + 4t^2 = 13,$
$t^2 + 8 = -2st.$

209. Symmetric systems. A system of equations in x and y is **symmetric** if the system is not altered by substituting x for y and y for x.

Thus $x^2 + y^2 = 6$ and $x + y = 11$ is a symmetric system, but $x^2 - y^2 = 6$ and $x + y = 11$ is not.

Certain symmetric systems or systems which are nearly symmetric can be easily solved by the method of substitution. Of such the following are types:

$$A\begin{cases} x \pm y = 6, \\ xy = 4. \end{cases} \qquad B\begin{cases} x^2 + y^2 = 25, \\ x \pm y = 2. \end{cases} \qquad C\begin{cases} 4x^2 + y^2 = 13, \\ 2x \pm y = 11. \end{cases}$$

A few other systems which are symmetric or nearly so are more easily solved by certain special methods. The following list contains typical systems, and the methods applicable are given in Exercises 1, 10, and 12.

EXERCISES

1. Solve $\begin{cases} x^2 + y^2 = 37, & (1) \\ xy = 6. & (2) \end{cases}$

Hints. These equations can be combined in such a way as to obtain definite values for $x + y$ and $x - y$ as follows:

(2) · 2, $\quad 2xy = 12.$ (3)

(1) + (3), $\quad x^2 + 2xy + y^2 = 49.$ (4)

From (4), $\quad x + y = \pm 7.$ (5)

(1) − (3), $\quad x^2 - 2xy + y^2 = 25.$ (6)

From (6), $\quad x - y = \pm 5.$ (7)

(5) and (7) combined give four systems of equations:

$$A\begin{cases} x + y = 7, & (8) \\ x - y = 5. & (9) \end{cases} \qquad C\begin{cases} x + y = -7, & (11) \\ x - y = 5. & (9) \end{cases}$$

$$B\begin{cases} x + y = 7, & (8) \\ x - y = -5. & (10) \end{cases} \qquad D\begin{cases} x + y = -7, & (11) \\ x - y = -5. & (10) \end{cases}$$

The solution of systems A, B, C, and D is left to the student.

The pairs of roots found for the four systems A, B, C, and D will check in the original system.

2. $x^2 + y^2 = 100,$ $\quad xy = 48.$

3. $x^2 + xy + y^2 = 19,$ $\quad xy = 6.$

4. $x^2 + 4y^2 = 20,$ $\quad xy = 4.$

5. $9x^2 + y^2 = 61,$
$xy = 10.$

6. $4x^2 + xy + y^2 = 85,$
$xy = 12.$

7. $4x^2 + 25y^2 = 41,$
$xy = 2.$

8. $x^2 - xy + y^2 = 8,$
$xy + 1 = 0.$

9. $\frac{x^2}{3} - xy + \frac{4y^2}{3} = \frac{7}{3},$
$x = \frac{6}{y}.$

10. $\frac{1}{x^2} + \frac{1}{y^2} = 13,$
$\frac{1}{xy} = 6.$

Hint. To clear of fractions in Exercise 10 would merely increase the difficulty of solution. Instead we solve in a manner similar to that of Exercise 1 (see Exercise 13, p. 225).

$$\frac{2}{xy} = 12.$$

Then $$\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = 25.$$

Whence $$\frac{1}{x} + \frac{1}{y} = \pm 5.$$

In like manner $$\frac{1}{x} - \frac{1}{y} = \pm 1.$$

11. $\frac{1}{x^2} + \frac{1}{y^2} = 25,$
$\frac{1}{xy} = 12.$

12. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{4},$
$\frac{1}{x} - \frac{1}{y} = \frac{1}{2}.$

Hints. $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = \frac{1}{4}.$

Then $$\frac{2}{xy} = 1, \text{ etc.}$$

13. $\frac{1}{3x} + \frac{1}{2y} = \frac{13}{6},$
$\frac{1}{9x^2} + \frac{1}{4y^2} = \frac{97}{36}.$

14. $\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2} = 7,$
$\frac{1}{xy} = 6.$

15. $\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2} = 13,$
$\frac{1}{x} + \frac{1}{y} = 7.$

210. Equivalent systems. **Equivalent systems** of equations are systems which have the same set or sets of roots.

If the two systems

$$A\begin{cases} x^2 - y^2 = 12, & (1) \\ x + y = 6. & (2) \end{cases} \quad \text{and} \quad B\begin{cases} x - y = 2, & (3) \\ x + y = 6. & (4) \end{cases}$$

are solved, only one pair of roots, $x = 4$ and $y = 2$, is obtained for A and the same pair for B. Systems A and B are equivalent, though usually a system which consists of a linear and a quadratic has two sets of roots and hence cannot be equivalent to a linear system (see page 411, second paragraph).

Sometimes an equation simpler than either of those given in a system can be derived from this system by dividing the left and right members of the first equation by the corresponding members of the second. In systems like those of the following list the equation so obtained taken with one of the first two gives an equivalent system more easily solved than the original one.

EXERCISES

Solve, using division where possible, pair results, and check each set of real roots:

1. $x^2 - 4y^2 = 20,$
   $x + 2y = 2.$

   HINT. Division gives the equivalent system

   $$x - 2y = 10,$$
   $$x + 2y = 2.$$

2. $x^2 - y^2 = 77,$
   $x - y = 7.$

3. $x^2 - 2xy = 187,$
   $2y - x = -17.$

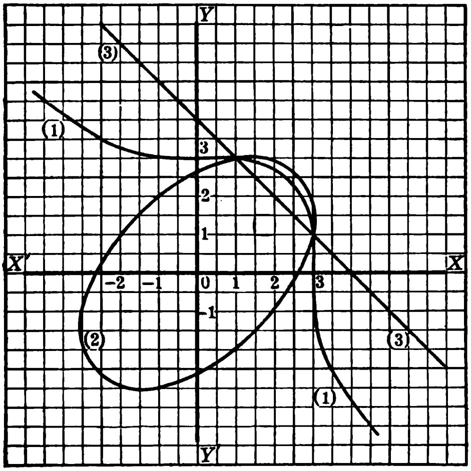
4. $4x^2 - y^2 = 16,$
   $2x + y = 8.$

5. $R^2h - 75 = 0,$
   $Rh = 15.$

6. $\frac{1}{x^2} - \frac{1}{y^2} = 15,$
   $\frac{1}{x} + \frac{1}{y} = 3.$

7. $x^3 + y^3 = 28,$
   $x + y = 4.$

In the following figure, (1), (2), and (3) are the graphs of $x^3 + y^3 = 28$, $x^2 - xy + y^2 = 7$, and $x + y = 4$ respectively. These equations are all used in the solution of Exercise 7. The graph makes clear in a striking way that the system (1) and (3) is equivalent to the system (2) and (3).



8. $x^3 - 8y^3 = 35,$
$x - 2y = 5.$

9. $\frac{1}{x} + \frac{1}{y} = 1,$
$\frac{1}{y^2} - \frac{1}{x^2} = 5.$

10. $\frac{1}{x^3} + \frac{1}{y^3} = 152,$
$\frac{1}{x} + \frac{1}{y} = 8.$

MISCELLANEOUS EXERCISES

Solve by any method and pair results. If any system cannot be solved algebraically by the methods previously given, solve it graphically.

1. $y^2 + xy = 8,$
$x + y = 4.$

2. $x^2 - 9y^2 = 7,$
$x - 3y = 1.$

3. $x^2 - x - 2y = 0,$
$y^2 - 4y + x = 0.$

4. $x^2 + y^2 = 25,$
$x + y = 7.$

5. $\frac{x}{y} + 3 = 0,$
$xy + 12 = 0.$

6. $(x + y)^2 = 9,$
$(x - y)^2 = 49.$

7. $2x^2 + y^2 = 33,$
$x^2 + 2y^2 = 54.$

8. $3h^2 - 8k^2 = 40,$
$5h^2 + k^2 = 81.$

9. $4R_1^2 + 3 = 9R_2^2,$
$12R_1^2 + R_2^2 = \frac{31}{9}.$

10. $xy + x = 18,$
$xy + y = 20.$

11. $x^2 + y^2 = 169,$
$xy = 60.$

12. $x^2 = y,$
$xy = 8.$

13. $x - xy = 5,$
$2y + xy = 6.$

14. $x^3 - y^3 = 19,$
$x^2 + xy + y^2 = 19.$

15. $4n^2 + 7m^2 = 9,$
$2n^2 - \frac{9}{2} = m^2.$

16. $5W_1^2 - 6.8W_2^2 = 99.55,$
$W_1^2 - W_2^2 = 20.$

17. $xy + 2y^2 = 2,$
$3xy + 5y^2 = 2.$

18. $x^2 + 2xy + 2y^2 = 10,$
$3x^2 - xy - y^2 = 51.$

19. $y^2 + x = 7,$
$x^2 + y = 11.$

20. $x^2 + xy + y^2 = 7,$
$x^2 + y^2 = 10.$

21. $x^2 + xy + y = 0,$
$x^2 + xy + x = 0.$

22. $\frac{1}{x^2} + \frac{1}{y^2} = 13,$
$\frac{1}{x} - \frac{1}{y} = 1.$

23. $\frac{1}{x^3} - \frac{1}{y^3} = 7,$
$\frac{1}{x} - \frac{1}{y} = 1.$

24. $4x^2 + y^2 = 289,$
$xy = 60.$

25. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 9,$
$x^{\frac{1}{2}}y^{\frac{1}{2}} = 20.$

26. $x^2 + y^2 + x + y = 58,$
$x + y = 5.$

27. $9x \div y = 18 = xy.$

28. $4x + \frac{1}{y} = 46 = \frac{26x}{5} - \frac{1}{y}.$

29. $\frac{1}{x-2} + \frac{1}{y-2} = \frac{3}{4},$
$\frac{1}{x} - \frac{1}{y} = \frac{1}{12}.$

30. $$\begin{aligned}3xy &= x^2y^2 - 88,\\ x - y &= 6.\end{aligned}$$

31. $$\begin{aligned}xy &= c,\\ x + y &= a.\end{aligned}$$

32. $$\begin{aligned}x^{-2} - y^{-2} &= 6,\\ x^{-1} + y^{-1} &= 2.\end{aligned}$$

33. $$\begin{aligned}x - y &= 16,\\ x^{\frac{1}{2}} - y^{\frac{1}{2}} &= 2.\end{aligned}$$

34. $$\begin{aligned}\frac{x-1}{y-1} &= 3,\\ \frac{y^2+y+1}{x^2-x+1} &= \frac{13}{43}.\end{aligned}$$

35. $$\begin{aligned}4x^2 - 13xy + 9y^2 &= 9,\\ xy - y^2 &= 3.\end{aligned}$$

36. $$\begin{aligned}x^2 + 2xy &= 16,\\ 3x^2 - 4xy + 2y^2 &= 6.\end{aligned}$$

37. $$\begin{aligned}x^3 &= y^3 + 37,\\ x^2y &= xy^2 + 12.\end{aligned}$$

PROBLEMS

(Reject all results which do not satisfy the conditions of the problems.)

1. Find two numbers whose difference is 6 and the difference of whose squares is 120.

2. The sum of two numbers is 20 and the sum of their squares is 202. Find the numbers.

3. Find two numbers whose sum plus their product is 132 and whose quotient is 3.

4. It takes 56 rods of fence to inclose a rectangular lot whose area is one acre. Find the dimensions of the lot.

5. The area of a right triangle is 180 square feet and its hypotenuse is 41 feet. Find the legs.

6. The area of a pasture containing 15 acres is doubled by increasing its length and its breadth by 20 rods. What were the dimensions at first?

7. The difference of the areas of two squares is 495 square feet and the difference of their perimeters is 60 feet. Find a side of each square.

8. The area of a rectangular field is $43\frac{1}{5}$ acres and one diagonal is 120 rods. Find the perimeter of the field.

9. The value of a certain fraction is $\frac{2}{3}$. If the fraction is squared and 44 is subtracted from both the numerator and the denominator of this result, the value of the fraction thus formed is $\frac{5}{14}$. Find the original fraction.

10. Two men together can do a piece of work in $4\frac{4}{5}$ days. One man requires 4 days less than the other to do the work alone. Find the number of days each requires alone.

11. The perimeter of a rectangle is 250 feet and its area is 214 square yards. Find the length and the width.

12. The base of a triangle is 8 inches longer than its altitude and the area is $1\frac{2}{3}$ square feet. Find the base and altitude of the triangle.

13. The volumes of two cubes differ by 316 cubic inches and their edges differ by 4 inches. Find the edge of each.

14. The perimeter of a right triangle is 80 feet and its area is 240 square feet. Find the legs and the hypotenuse.

15. The perimeter of a rectangle is $7a$ and its area is a^2. Find its dimensions.

16. A man travels from A to B, 30 miles, by boat and from B to C, 120 miles, by rail. The trip required 6 hours. He returned from C to B by a train running 10 miles per hour faster, and from B to A by the same boat. The return trip took 5 hours. Find the rate of the boat and of each train.

17. There were 1400 fewer reserved seats at a certain sale than of unreserved seats, and the price of the latter was 15 cents less than the price of the former. The total proceeds were $490, of which $250 came from the reserved seats. Find the number of each kind of seats and the price of each.

18. If a two-digit number be multiplied by the sum of its digits, the product is 324; and if three times the sum of its digits be added to the number, the result is expressed by the digits in reverse order. Find the number.

19. The sum of the radii of two circles is 31 inches and the difference of their areas is 155π square inches. Find the radii.

20. The yearly interest on a certain sum of money is $42. If the sum were $200 more and the interest 1% less, the annual income would be $6 more. Find the principal and the rate.

21. A wheelman leaves A and travels north. At the same time a second wheelman leaves a point 3 miles east of A and travels east. One and one-third hours after starting, the shortest distance between them is 17 miles, and $3\frac{1}{3}$ hours later the distance is 53 miles. Find the rate of each.

22. A starts out from P to Q at the same time B leaves Q for P. When they meet, A has gone 40 miles more than B. A then finishes the journey to Q in 2 hours and B the journey to P in 8 hours. Find the rates of A and B and the distance from P to Q.

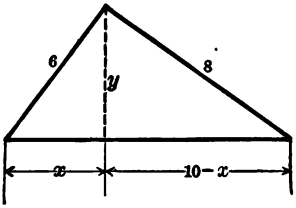
23. A leaves P going to Q at the same time that B leaves Q on his way to P. From the time the two meet, it requires $6\frac{2}{3}$ hours for A to reach Q, and 15 hours for B to reach P. Find the rate of each, if the distance from P to Q is 300 miles.

GEOMETRICAL PROBLEMS

1. The sides of a triangle are 6, 8, and 10. Find the altitude on the side 10.

HINT. From the accompanying figure we easily obtain the system

$$\begin{cases} x^2 + y^2 = 36, \\ (10 - x)^2 + y^2 = 64. \end{cases}$$



2. The sides of a triangle are 8, 15, and 17. Find the altitude of the triangle on the side 17 and the area of the triangle.

3. The sides of a triangle are 11, 13, and 20. Find the altitude on the side 20 and the area of the triangle.

4. The sides of a triangle are 13, 14, and 15. Find the altitude on the side 14 and the area of the triangle.

5. The sides of a triangle are 12, 17, and 25. Find the altitude on the side 12 and the area of the triangle.

6. Find correct to two decimals the altitude on the side 16 of a triangle whose sides are 16, 20, and 24 respectively.

7. The parallel sides of a trapezoid are 14 and 26 respectively, and the two nonparallel sides are 10 each. Find the altitude of the trapezoid.

HINT. Let $ABCD$ be a trapezoid. Draw CE parallel to DA and CF perpendicular to AB.

Then

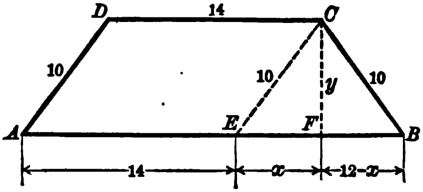
$$EC = 10,$$

and

$$AE = 14,$$

and

$$EB = 26 - 14, \text{ or } 12.$$



If we let $EF = x$, FB must equal $12 - x$; then we can obtain the system of equations

$$\begin{cases} x^2 + y^2 = 100, \\ (12 - x)^2 + y^2 = 100. \end{cases}$$

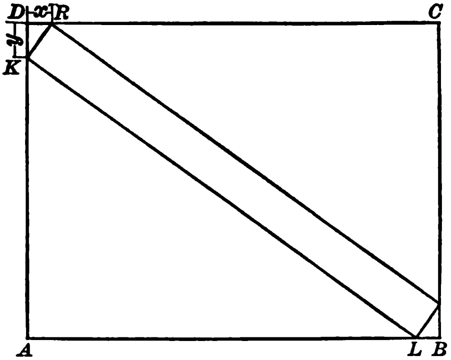
8. The two nonparallel sides of a trapezoid are 10 and 17 respectively, and the two bases are 9 and 30 respectively. Find the altitude of the trapezoid.

9. The bases of a trapezoid are 15 and 20 respectively, and the two nonparallel sides are 29 and 30. Find the altitude of the trapezoid and the area.

10. The sides of a trapezoid are 7, 9, 20, and 24. The sides 24 and 9 are the bases. Find the altitude and the area.

11. The sides of a trapezoid are 21, 27, 40, and 30. The sides 21 and 40 are parallel. Find the altitude and the area of the trapezoid.

12. The sides of a trapezoid are 23, 85, 100, and x. The sides 23 and 100 are the bases, and each is perpendicular to the side x. Find x and the area of the trapezoid.



13. The area of a triangle is 1 square foot. The altitude on the first side is 16 inches. The second side is 14 inches longer than the third. Find the three sides.

14. A rectangular tank is 8 feet 6 inches long and 6 feet 8 inches wide. A board 10 inches wide is laid diagonally on the floor. What two equations must be solved to determine the length of the longest board that can be thus laid?

HINTS. Let $DR = x$ and $DK = y$. The triangle DKR is similar to the triangle AKL.

CHAPTER XXXII

PROGRESSIONS

211. A sequence of numbers. In all fields of mathematics we frequently encounter groups of three or more numbers, selected according to some law and arranged in a definite order, whose relations to each other and to other numbers we wish to study.

There is an unlimited variety of such groups, or successions, of numbers. Only two simple types will be considered here.

212. Arithmetical progression. An **arithmetical progression** is a succession of terms in which each term after the first is formed by adding the same number to the preceding one.

Thus, if a denotes the first term and d the common number added, any arithmetical progression is represented by

$$a, a + d, a + 2d, a + 3d, a + 4d, \cdots.$$

This common number d is called the **common difference** and may be any number, *positive* or *negative.* It may be found for any given arithmetical progression by subtracting any term from the term which follows it.

The numbers 3, 7, 11, 15, $\cdots$ form an arithmetical progression, since any term, after the first, minus the preceding one gives 4. Similarly, 12, 6, 0, -6, $-12, \cdots$ is an arithmetical progression, since any term, after the first, minus the preceding one gives the common difference -6. In like manner $\frac{7}{2}$, 5, $6\frac{1}{2}, \cdots$ is an arithmetical progression whose common difference is $1\frac{1}{2}$.

ORAL EXERCISES

State the first four terms of the arithmetical progression if:

1. $a = 2$, $d = 3$.

2. $a = 5$, $d = 4$.

3. $a = 10$, $d = 6$.

4. $a = 20$, $d = 5$.

5. $a = 18$, $d = -3$.

6. $a = 100$, $d = -10$.

7. $a = 20x$, $d = -2x$.

8. $a = x$, $d = 2x$.

9. $a = 17m$, $d = -2m$.

10. $a = \sqrt{5}$, $d = 1 + \sqrt{5}$.

From the following select the arithmetical progressions, and in each arithmetical progression find the common difference:

11. $1, 10, 19, \cdots$.

12. $4, 12, 36, \cdots$.

13. $19, 11, 3, \cdots$.

14. $9, 12, \frac{1}{2}, 16, \cdots$.

15. $3, \frac{20}{8}, \frac{31}{8}, \cdots$.

16. $8, 9\frac{2}{5}, 10\frac{4}{5}, \cdots$.

17. $15, 3, -12, \cdots$.

18. $6a, 10a, 14a, \cdots$.

19. $18a, 14a, 12a, \cdots$.

20. $5a, 5a + 3, 5a + 6, \cdots$.

21. $3x, x - 2, -x - 4, \cdots$.

22. $2x - 1, x, 1, 2 - x, \cdots$.

23. $-5\sqrt{a}, -2\sqrt{a}, \sqrt{a}, 4\sqrt{a}, \cdots$.

24. $5\sqrt{x} - 1, 4\sqrt{x} - 2, 3\sqrt{x} - 3, \cdots$.

213. The last or *n*th term of an arithmetical progression. In the arithmetical progression

$$a,\ a + d,\ a + 2d,\ a + 3d,\ a + 4d,\ \cdots$$

one observes that the coefficient of d in each term is 1 less than the number of the term. Hence the nth, or general, term is $a + (n - 1)d$. If l denotes the nth term, we have

$$l = a + (n - 1)d.$$

difference [illegible] term [illegible] is [illegible] term [illegible] term

EXERCISES

Find the required terms of the following arithmetical progressions:

1. The fifteenth term of 2, 7, 12, $\cdots$.

Solution. Here $a = 2$, $d = 5$, and $n = 15$.

$a + (n-1)d$

Hence $$l = a + (n-1)d$$

becomes $$l = 2 + 14 \cdot 5 = 72.$$

2. The eighth term of 1, 4, 7, $\cdots$.

3. The eleventh term of 15, 9, 3, -3, $\cdots$.

4. The twenty-first term of a, $4a$, $7a$, $\cdots$.

5. The sixteenth term of $9x$, x, $-7x$, $\cdots$.

6. The sixth term of $\frac{10}{3}$, $\frac{6}{3}$, $\frac{2}{3}$, $\cdots$.

7. The eleventh term of $-\frac{1}{7}$, $\frac{3}{7}$, 1, $\cdots$.

8. The sixth term of $\sqrt{3}$, $4\sqrt{3}$, $7\sqrt{3}$, $\cdots$.

9. The eighth term of $3\sqrt{5}$, $\sqrt{5}$, $-\sqrt{5}$, $-3\sqrt{5}$, $\cdots$.

10. The eighth term of $5 + a$, $7 + 3a$, $9 + 5a$, $\cdots$.

11. The twelfth and the twentieth terms of 0, $-5x + 2$, $4 - 10x$, $\cdots$.

12. The tenth term of $\sqrt{a} + 3$, $5\sqrt{a} + 7$, $9\sqrt{a} + 11$, $\cdots$.

13. The tenth and the twentieth terms of $9\sqrt{a} - 3$, $5\sqrt{a}$, $\sqrt{a} + 3$, $\cdots$.

14. The fourteenth term of $\frac{\sqrt{x}}{2}$, $\frac{5\sqrt{x}}{2}$, $\frac{9\sqrt{x}}{2}$, $\cdots$.

15. The ninth and the twelfth terms of $\sqrt{3}$, $\frac{6}{\sqrt{3}}$, $3\sqrt{3}$, $\cdots$.

16. The eighth and the sixteenth terms of $\frac{\sqrt{x}}{2} + 1$, $\frac{3\sqrt{x}}{2} + 2$, $\frac{5\sqrt{x}}{2} + 3$, $\cdots$.

17. The fifteenth term of $\frac{2\sqrt{a}}{3} - 12$, -5, $\frac{2(3 - \sqrt{a})}{3}$, $\cdots$.

Be able to find formula 14 33

18. The twenty-ninth term of $a, a + d, a + 2d, \cdots$.

19. The mth term of $a, a + d, a + 2d, \cdots$.

20. The $(m - 1)$th term of $a, a + d, a + 2d, \cdots$. 434

21. The $(n - 2)$th term of $a, a + 5, a + 10, \cdots$.

22. The $(n - 5)$th term of $\frac{3}{\sqrt{3}}, 3\sqrt{3}, 5\sqrt{3}, \cdots$.

23. Find the $(n-3)$th term of $\sqrt{5}-1, 2\sqrt{5}-2, 3(\sqrt{5}-1), \cdots$.

24. Find the nth term of $\frac{1}{a}, \frac{a-1}{a}, 2 - \frac{3}{a}, \cdots$.

25. The first and third terms of an arithmetical progression are 2 and 22. Find the seventh term; the nth term.

26. The first and second terms of an arithmetical progression are r and s respectively. Find the third term and the nth term.

27. The edges of a box are consecutive even integers with n the least. Express in terms of n, (a) the sum of the edges; (b) the area of the faces; (c) the volume.

28. A body falls 16 feet the first second, 48 feet the next, 80 feet the next, and so on. How far does it fall during the twelfth second? during the nth second?

29. The digits of a certain three-digit number are in arithmetical progression. If their sum is 24 and the sum of their squares is 194, find the number.

214. Arithmetical means. The **arithmetical means** between two numbers are numbers which form, with the two given ones as the first and the last term, an arithmetical progression.

The insertion of one or more arithmetical means between two numbers is performed as in the following examples:

EXAMPLES

1. Insert four arithmetical means between 7 and 72.

Solution. $l = a + (n - 1)d$.

There will be six terms in all.

Therefore $\qquad 72 = 7 + (6 - 1)d$.

Solving, $\qquad d = 13$.

The required arithmetical progression is 7, 20, 33, 46, 59, 72.

2. Insert one arithmetical mean between h and k.

Solution. $l = a + (n - 1)d$.

There will be three terms in all.

Therefore $\qquad k = h + 2d$.

Solving, $\qquad d = \frac{k - h}{2}$.

Therefore the progression is h, $h + \frac{k - h}{2}$, k, or h, $\frac{h + k}{2}$, k.

It follows from the above that the arithmetical mean between two numbers is one half of their sum.

EXERCISES

Insert one arithmetical mean between:

1. 3 and 15. **2.** 3 and 27. **3.** h and $5h$. **4.** m and n.

Insert two arithmetical means between:

5. 2 and 8. **7.** 19 and 217. **9.** x and $3y$.

6. 5 and 29. **8.** x and $x + 6y$. **10.** h and k.

Insert three arithmetical means between:

11. 10 and 34. **13.** $-12x$ and $44x$.

12. -5 and 31. **14.** h and k.

15. Insert seven arithmetical means between -13 and 131.

16. Insert five arithmetical means between $-\frac{8}{3}$ and $\frac{46}{3}$.

17. Insert four arithmetical means between $-2\sqrt{5}$ and $18\sqrt{5}$.

18. Insert five arithmetical means between $7a - 3b$ and $13a + 9b$.

19. Insert six arithmetical means between $\sqrt{3}$ and $\frac{27}{2\sqrt{3}}$.

20. Insert two arithmetical means between $\frac{3}{1-\sqrt{3}}$ and $4\sqrt{3} - 6$.

21. In going a distance of 1 mile an engine increased its speed uniformly from 15 miles per hour to 25 miles per hour. What was the average velocity in miles per hour during that time? How long did it require to run the mile? 439

22. At the beginning of the third second the velocity of a falling body is 64 feet per second, and at the end of the third second it is 96 feet per second. What is its average velocity in feet per second during the third second? How many feet does it fall during the third second?

23. The velocity of a body falling from rest increases uniformly and is 32 feet per second at the end of the first second. What is the average velocity in feet per second during the first second? How many feet does the body fall during the first second? the second second?

24. Find the average length of twenty lines whose lengths in inches are the first twenty odd numbers.

25. Find the average length of fifteen lines whose lengths in inches are given by the consecutive even numbers beginning with 58.

26. With the conditions of Problem 23 determine the average velocity per second of a body which has fallen for 12 seconds.

27. A certain distance is separated into ten intervals, the lengths of which are in arithmetical progression. If the shortest interval is 1 inch and the longest 37 inches, find the others.

215. Sum of a series. The indicated sum of several terms of an arithmetical progression is called an arithmetical series. The formula for the sum of n terms of an **arithmetical series** may be obtained as follows:

$$S=a+(a+d)+(a+2d)+\cdots+(l-2d)+(l-d)+l. \quad (1)$$

Reversing the order of the terms in the second member of (1),

$$S=l+(l-d)+(l-2d)+\cdots+(a+2d)+(a+d)+a. \quad (2)$$

Adding (1) and (2),

$$2S=(a+l)+(a+l)+(a+l)+\cdots+(a+l)+(a+l)+(a+l)=n(a+l).$$

Therefore $$S=\frac{n}{2}(a+l).$$

EXAMPLE

Required the sum of the integers from 7 to 92, inclusive.

Solution. $n=86$, $a=7$, $l=92$.

Substituting in $S=\frac{n}{2}(a+l)$,

$$S=\frac{86(7+92)}{2}=4257.$$

Therefore the sum of the integers from 7 to 92 is 4257.

ORAL EXERCISES

Using the formula $S=\frac{n}{2}(a+l)$, find the sum of the following:

1. The six-term series in which $a=2$ and $l=17$.
2. The ten-term series in which $a=1$ and $l=46$.
3. The twelve-term series in which $a=-12$ and $l=21$.
4. The seven-term series in which $a=3$ and $l=63$.
5. The twenty-term series in which $a=5$ and $l=85$.
6. The twelve-term series in which $a=-175$ and $l=125$.

The formula $S = \frac{n}{2}(a + l)$ enables one to find the sum of a series if the first term, the last term, and the number of the terms are given. If, however, the last term is not given, but instead the common difference is given or evident, we can use a formula obtained by substituting in $S = \frac{n}{2}(a + l)$ the value of l from the formula $\underline{l = a + (n - 1)\,d}$, on page 433.

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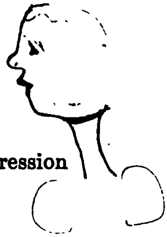
Then $$S = \frac{n}{2}[a + a + (n - 1)d],$$

or $$S = \frac{n}{2}[2a + (n - 1)d].$$



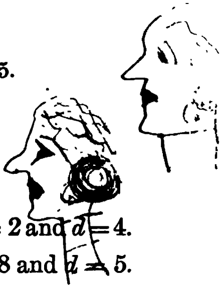
EXAMPLE

Required the sum of the first fifty-nine terms of the progression 2, 9, 16, ….

Solution. $n = 59$, $a = 2$, and $d = 7$.

Substituting in $S = \frac{n}{2}[2\,a + (n - 1)\,d]$,

$$S = \frac{59}{2}(4 + 58 \cdot 7) = \frac{59}{2}(410) = 12{,}095.$$



l each time [redacted]

Find the sum of the following:

1. The first eight terms of the series in which $a = 2$ and $d = 4$.

2. The first ten terms of the series in which $a = 8$ and $d = 5$.

3. The first nine terms of the series in which $a = 1$ and $d = 11$.

4. The first fifteen terms of the series in which $a = 17$ and $d =$ [illegible].

5. Th[illegible]irst twenty terms of the series in which $a = 100$ and $d = -$[illegible]

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6. The first eight terms of the series $2 + 4 + 6 + \cdots$.

7. The first ten terms of the series $1 + 9 + 17 + \cdots$.

8. The first ten terms of the series $-8 + (-4) + 0 + \cdots$.

9. The first eighteen terms of the series $1 + 5 + 9 + \cdots$.

10. The first twenty terms of the series $10 + 8 + 6 + \cdots$.

11. The first twelve terms of $1 + \frac{3}{2} + 2, \cdots$.

12. The first twelve terms of $15 + 12\frac{1}{2} + 10 + \cdots$.

13. The first one hundred integers.

14. The first one hundred even numbers.

15. The first one hundred odd numbers.

16. Show that the sum of the first n even numbers is $n(n+1)$.

17. Show that the sum of the first n odd numbers is n^2.

18. Find the sum of the even numbers between 247 and 539.

19. How many of the positive integers beginning with 1 are required to make their sum 861?

HINT. Substitute in the formula $S = \frac{n}{2}[2a + (n-1)d]$ and solve for n.

20. How many terms must constitute the series $7 + 10 + 13 + \cdots$ in order that its sum may be 242?

21. Beginning with 90 in the progression 78, 80, 82, $\cdots$, how many terms are required to give a sum of 372?

22. The second term of an arithmetical progression is -2 and the eighth term is 22. Find the eleventh term.

23. Find the sum of the first t terms of $\frac{1}{a}, \frac{a-1}{a}, \frac{2a-3}{a}, \cdots$.

24. If $l = 25$, $a = 1$, and $d = 4$, find n and s.

25. If $a = -20$, $d = 11$, and $s = 216$, find n and l.

26. If $d = -9$, $n = 15$, and $s = 0$, find a and l.

27. The first and second terms of an arithmetical progression are h and k respectively. Find the sum of n terms.

28. If $s = 9h$, $a = 12 - 10h$, and $n = 9$, find l and d.

29. If $s = 66\sqrt{3}$, $a = -4\sqrt{3}$, and $d = 2\sqrt{3}$, find n and l.

30. A clock strikes the hours but not the half hours. How many times does it strike in a day?

31. A car running 15 miles an hour is started up an incline, which decreases its velocity $\frac{1}{2}$ of a foot per second. (*a*) In how many seconds will it stop? (*b*) How far will it go up the incline?

32. A car starts down a grade and moves 3 inches the first second, 11 inches the second second, 19 inches the third second, and so on. (*a*) How fast does it move in feet per second at the end of the thirtieth second? (*b*) How far has it moved in the thirty seconds?

33. An elastic ball falls from a height of 24 inches. On each rebound it comes to a point $\frac{1}{2}$ inch below the height reached the time before. How often will it drop before coming to rest? Find the total distance through which it has moved.

34. The digits of a three-digit number are in arithmetical progression. The first digit is 3 and the number is $20\frac{1}{2}$ times the sum of its digits. Find the number.

35. A clerk received $60 a month for the first year and a yearly increase of $75 for the next nine years. Find his salary for the tenth year and the total amount received.

36. If a man saves $200 a year and at the end of each year places this sum at simple interest at 6%, what will be the amount of his savings at the time of the sixth annual deposit?

37. Assuming that a ball is not retarded by the air, determine the number of seconds it will take to reach the ground if dropped from the top of the Washington Monument, which is 555 feet high. With what velocity will it strike the ground?

HINT. See Exercise 28, p. 435.

38. A ball thrown vertically upward rose to a height of 256 feet. In how many seconds did it begin to fall? With what velocity was it thrown?

39. By Exercise 28, p. 435, it is seen that a falling body obeys the law of an arithmetical progression. Show from the data of that exercise that the general formula $S = \frac{n}{2}(2a + (n-1)d)$ becomes the special one $S = \frac{gt^2}{2}$, which is used in physics for all such problems.

40. A ball thrown vertically upward returned to the ground 6 seconds later. How high did it rise? With what velocity was it thrown?

41. A and B start from the same place at the same time and travel in the same direction. A travels 20 miles per hour. B goes 30 miles the first hour, 26 miles the second, 22 miles the third, and so on. When are they together?

NOTE. In the earliest mathematical work known a problem is found which involves the idea of an arithmetical progression. In the papyrus of the Egyptian priest Ahmes, who lived nearly two thousand years before Christ, we read in essence, "Divide 40 loaves among 5 persons so that the numbers of loaves that they receive shall form an arithmetical progression, and so that the two who receive the least bread shall together have one seventh as much as the others." From that time to this, the subject has been a favorite one with mathematical writers, and has been extended so widely that it would require many volumes to record all of the discoveries regarding the various kinds of series.

216. Geometrical progression. A **geometrical progression** is a succession of terms in which each term after the first is formed by multiplying the preceding one always by the same number.

Thus, if a denotes the first term and r the common multiplier, then any geometrical progression is represented by $a, ar, ar^2, ar^3, ar^4, \cdots$.

The common multiplier is called the *ratio.* It is evident from the above that the ratio r in a geometrical progression is found by dividing any term by the preceding one.

The numbers 2, 10, 50, 250, $\cdots$ form a geometrical progression, since any term, after the first, divided by the preceding one gives the same number 5. Similarly, the numbers $3, -3\sqrt{2}, 6, -6\sqrt{2}, \cdots$ form a geometrical progression, since any term, after the first, divided by the preceding one gives the common ratio $-\sqrt{2}$.

ORAL EXERCISES

Determine which of the following are geometrical progressions and state the ratio in each case:

1. $1, 3, 9, 27, \cdots$.

2. $2, 4, 16, \cdots$.

3. $2, 6, 18, \cdots$.

4. $5, 1, \frac{1}{5}, \cdots$.

5. $18, -3, \frac{1}{2}, \cdots$.

6. $2, \frac{1}{2}\sqrt{2}, \frac{1}{4}\sqrt{2}, \cdots$.

7. $\frac{1}{3}\sqrt{6}, \sqrt{6}, \sqrt{54}, \cdots$.

8. $\sqrt{\frac{2}{4}}, -\sqrt{\frac{3}{2}}, 2, \cdots$.

9. $7a, 35a, 175a, \cdots$.

10. $8\sqrt{5}, -2\sqrt{5}, \sqrt{5}, \cdots$.

11. $5x^2, 10x^2, 20x^2, \cdots$.

12. $3y^{\frac{1}{2}}, 12y, 48y^{\frac{3}{2}}, \cdots$.

13. Find the condition under which a, b, and c form a geometrical progression.

State in order the first four terms of the geometrical progression in which the first term is

14. 1 and the ratio is 4.

15. 3 and the ratio is 10.

16. -3 and the ratio is 2.

17. 64 and the ratio is $\frac{1}{2}$.

18. -243 and the ratio is $\frac{1}{3}$.

19. 2 and the ratio is $\sqrt{3}$.

217. The *n*th term of a geometrical progression. Since by definition any geometrical progression is represented by

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,$$

it is evident that the exponent of r in any term is 1 less than the number of the term. Therefore, if t_n denotes the *n*th, or general, term of any geometrical progression,

$$t_n = ar^{n-1}.$$

EXERCISES

1. Find the fifth term of 4, 12, 36, ...

Solution. Here $a = 4$, $r = 3$, $n - 1 = 4$.
Substituting these values in the formula $t_n = ar^{n-1}$,

$$t_4 = 4 \cdot 3^4 = 324.$$

2. Find the eighth term of 1, 2, 4, ...

3. Find the tenth term of 3, 6, 12, ...

4. Find the eighth term of 3, 2, $\frac{4}{3}$, ...

5. Find the twelfth term of 7, −14, 28, ...

6. Find t_9 of 15, −5, $+\frac{5}{3}$, ...

7. Find t_{12} of $12a^2$, $9a^2$, $\frac{27a^2}{4}$, ...

8. Find t_6 of $\frac{-2c^3}{5}$, -1, $\frac{-5}{2c^3}$, ...

9. Find t_8 of $4\sqrt{2}$, 4, $2\sqrt{2}$, ...

10. Find t_7 of $\frac{1}{2\sqrt{2}}$, $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, ...

11. Write the twentieth term of \$4.12(1.01), \$4.12(1.01)^2, \$4.12(1.01)^3, ...

12. Since the nth term of a geometrical progression is ar^{n-1}, what is the $(n-1)$th term? the $(n-2)$th? the $(n-3)$th? the $(n+1)$th? the $(n+2)$th?

13. The first and second terms of a geometrical progression are m and n respectively. Find the next two terms.

218. Geometrical means. **Geometrical means** between two numbers are numbers which form, with the two given ones as the first and the last term, a geometrical progression.

Thus ar and ar^2 are the geometrical means in a, ar, ar^2, ar^3.

EXAMPLES

1. Insert two real geometrical means between 9 and 72.

Solution. There are four terms in the geometrical progression, and $a = 9$, $n = 4$, and $t_n = t_4 = 72$.

Substituting these values in $t_n = ar^{n-1}$,

$$72 = 9r^3.$$

4

Whence $r = 2$, and $-1 \pm \sqrt{-3}$.

The required geometrical progression is 9, 18, 36, 72.

4

2. Insert one geometrical mean between h and k.

Solution. There are three terms in the progression, and $a = h$, $n = 3$, and $t_n = k$.

Substituting these values in $t_n = ar^{n-1}$, we have

$$k = h \cdot r^2.$$

4

Solving, $r = \pm\sqrt{\frac{k}{h}}.$

Hence the progression is $k, \pm h\sqrt{\frac{k}{h}}, k$, or $h, \pm\sqrt{hk}, k$.

It follows from the above that the geometrical mean between two numbers is the square root of their product.

ORAL EXERCISES

Insert one geometrical mean between:

1. 1 and a^2.
2. 1 and x^4.
3. 1 and $4x^2$.
4. 3 and 75.
5. a and a^3.
6. a^5 and a^9.

Insert two geometrical means between:

7. 1 and x^3.
8. 1 and 2^3.
9. 1 and 27.
10. 1 and 125.
11. a and a^4.
12. x and x^7.
13. x^2 and x^8.
14. 2 and 16.
15. 5 and 40.

EXERCISES

Obtain progressions involving real terms only:

1. Insert two geometrical means between 21 and 168.

2. Insert two geometrical means between 15 and 405.

3. Insert three geometrical means between 3 and 243.

4. Insert one geometrical mean between 9 and 81.

5. Insert one geometrical mean between a^{12} and a^{20}.

6. Insert three geometrical means between -9 and -144.

7. The fourth term of a geometrical progression is 16, the eighth term is 256. Find the tenth term.

8. The second term of a geometrical progression is $4\sqrt{3}$, the fifth term is $\frac{4}{9}$. Find the first term and the ratio.

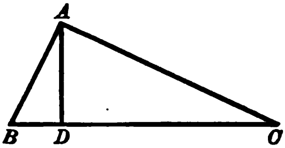
9. Show that the geometrical means between a and c are $\pm\sqrt{ac}$.

10. The first and fourth terms of a geometrical progression are a and c respectively. Find the second and third terms.

11. Insert three geometrical means between h and k.

12. The sum of the first and fourth terms of a geometrical progression is 56. The second term is 6. Find the four terms.

13. In the accompanying figure, ABC is a right triangle and AD is perpendicular to the hypotenuse BC. Under these conditions the length of AD is always a geometrical mean between the lengths of BD and DC.

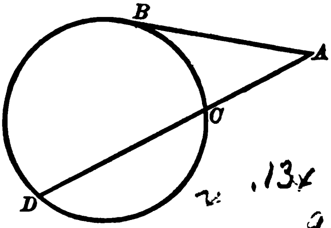


(*a*) If $BD = 4$ and $DC = 9$, find AD.

(*b*) If $BD = 3$ and $BC = 21$, find AD.

(*c*) If $BC = 25$ and $AD = 10$, find BD and DC.

14. In the accompanying figure, AB touches and AD cuts the circle. Under such conditions the length of AB is always a geometrical mean between the lengths of AC and AD.



.13x .036

9

4

(*a*) If $AD = 36$ and $AC = 4$, find AB.

(*b*) If $DC = 90$ and $AB = 24$, find AC and AD.

219. Geometrical series. The indicated sum of n terms of a geometrical progression is called a **geometrical series.** The process of obtaining in its simplest form the expression for this sum is often called finding the sum of the series.

The expression for the sum is derived as follows:

Let S_n denote the indicated sum.

$$S_n = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}$$
$$= a(1 + r + r^2 + \cdots + r^{n-3} + r^{n-2} + r^{n-1}). \qquad (1)$$

$$S_n = a\left(\frac{1-r^n}{1-r}\right), \qquad (2)$$

since the polynomial in (1) is equal to the fraction in the parenthesis of (2) (see Exercise 18, p. 304).

Hence $$S_n = \frac{a - ar^n}{1-r}.$$

EXERCISES

1. Find the sum of the first nine terms of $3, -6, 12, \cdots$.

Solution. $$S_n = \frac{a - ar^n}{1 - r}.$$

By the conditions, $a = 3$, $r = -2$, and $n = 9$.

Substituting, $$S_9 = \frac{3 - 3(-2)^9}{1 - (-2)} = 513.$$

2. Find the sum of $1, 5, 25, \cdots$ to nine terms.

3. Find S_8 for $2, -4, 8, \cdots$.

4. Find S_7 for $40, 20, 10, \cdots$.

5. Find S_8 for $-180, 90, -45, \cdots$.

6. Find S_6 for $\frac{2}{3}, 1, \frac{3}{2}, \cdots$.

7. Find S_{10} for $a^3, a^5, a^7, \cdots$.

8. Find S_7 for $2\sqrt{3}, 6, 6\sqrt{3}, \cdots$.

9. Find S_n for $4, 12, 36, \cdots$.

10. Find S_n for $125, -25\sqrt{5}, 25, \cdots$.

11. Find S_{n-1} for $3, 12, 48, \cdots$.

12. Find S_n for $3, -15, 75, \cdots$.

13. Find S_{n-2} for $x, 4x^4, 16x^7, \cdots$.

14. Show that for a geometrical progression $S_n = \dfrac{a - rl}{1 - r}$.

HINT. Substitute l for the factor ar^{n-1} in the last term of the numerator in the formula of Exercise 1.

15. A rubber ball falls from a height of 60 inches and on each rebound rises 50% of the previous height. How far does it fall on its sixth descent? Through what distance has it moved at the end of the sixth descent?

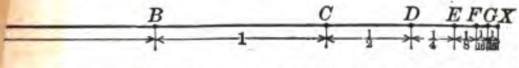
220. Infinite geometrical series. If the number of terms of a geometrical series is unlimited, it is called an **infinite geometrical series.**

In the progression 2, 4, 8, ⋯ the ratio is positive and greater than 1, and each term is greater than the term preceding it. Such a progression is said to be increasing. Obviously the sum of an unlimited number of terms in such an increasing geometrical progression is unlimited. If $r > 1$ (read "r is greater than 1"), the sum can be made as large as we please by taking enough terms.

In the progression 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, ⋯ the ratio is
positive and less than 1, and each term is less than the term
it. Such a progression is said to be decreasing.
ber of terms of such a geometrical pro-
ted, the sum of as many terms as we
lways less than some definite number.

rst three terms of the series $2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$
erms, is $3\frac{3}{4}$; of five terms, is $3\frac{7}{8}$; of six terms,
ns, is $3\frac{31}{32}$, etc. Here for any number of terms
ss than 4.

sum of the series $2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$
rence to the following diagram:



the above series are represented by intervals on the
is four units in length; that is, $AB=2$, $BC=1$,
$EF=\frac{1}{8}$, $FG=\frac{1}{16}$, etc.

e several intervals reveals the fact that as we pass
each new interval is just one half of the then unused
It follows, then, that the distance laid off gradually
ugh it never equals, AX, for at no time is an interval
is more than one half of that which then remained.
re will always be an unused interval between the
the last interval used and the point X.

um of the intervals laid off approaches but never quite
eq terval AX, we have in the above diagram an illustration
of the hat the sum of the series $2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$
approache yet always remains less than, 4.

The formula for the sum of a geometric series $S_n=\frac{a-ar^n}{1-r}$ may be written as the difference of two fractions,

$$S_n=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

For the series $2+1+\frac{1}{2}+\frac{1}{4}+\cdots$ it follows that

$$S_n=\frac{2}{1-\frac{1}{2}}-\frac{2(\frac{1}{2})^n}{1-\frac{1}{2}}. \qquad (3)$$

Now $(\frac{1}{2})^2=\frac{1}{4}$, $(\frac{1}{2})^3=\frac{1}{8}$, $(\frac{1}{2})^4=\frac{1}{16}$, $(\frac{1}{2})^5=\frac{1}{32}$. Consequently $(\frac{1}{2})^n$ becomes very small if n is taken very great. Therefore $2(\frac{1}{2})^n$, the numerator of the last fraction in (3), decreases and approaches zero as n increases *without limit.* Since the denominator of the fraction remains $\frac{1}{2}$ while the numerator approaches zero, the value of the fraction decreases and approaches zero as n increases. Then if S_∞ denotes S_n where n has increased without limit, we may write

$$S_\infty \text{ approaches } \frac{2}{1-\frac{1}{2}}, \text{ or } 4.$$

In general, if $r<1$ ("if r is numerically less than 1"), the numerical value of fraction $\frac{ar^n}{1-r}$ approaches zero as n increases without limit. Under such conditions the formula

$$S_n=\frac{a}{1-r}-\frac{ar^n}{1-r} \text{ becomes } S_\infty=\frac{a}{1-r}.$$

This means that for r numerically less than 1, S_n approaches $\frac{a}{1-r}$, but for any definite value of n it is always numerically less than this number.

Hence whenever we speak of the sum of such a series we mean the *limit* which the sum approaches as n increases indefinitely.

EXERCISES

Find the sum of the following series.

1. $4 + 1 + \frac{1}{4} + \cdots$.

Solution. $S_\infty = \dfrac{a}{1-r}$.

Substituting, $S_\infty = \dfrac{4}{1-\frac{1}{4}} = 5\frac{1}{3}$.

2. $1 + \frac{1}{2} + \frac{1}{4} + \cdots$.

3. $3 + (-1) + \frac{1}{3} + \cdots$.

4. $5 + (-2) + \frac{4}{5} + \cdots$.

5. $2 + \sqrt{2} + 1 + \cdots$.

6. $5a + \dfrac{5a}{2} + \dfrac{5a}{4} + \cdots$.

7. $1 + x + x^2 + \cdots$. $(x < 1)$

8. $5 + \sqrt{5} + 1 + \cdots$.

9. $1 + \dfrac{1}{x} + \dfrac{1}{x^2} + \cdots$. $(x > 1)$

10. .121212.

Hint. $.121212 = \frac{12}{100} + \frac{12}{10000} + \frac{12}{1000000} + \cdots$.

11. .666 ⋯.

12. .151515 ⋯.

13. .3232 ⋯.

14. 25.2727 ⋯.

15. .71515 ⋯.

16. .3108108 ⋯.

17. A flywheel whose circumference is 5 feet makes 80 revolutions per second. If it makes 99% as many revolutions each second thereafter as it did the preceding second, how far will a point on its rim have moved by the time it is about to stop?

Note. In the study of geometrical progressions we have seen that the sum of the infinite series $1 + x + x^2 + x^3 + \cdots$ is a definite number when x has any value less than 1. But it has no finite value when x is equal to or greater than 1; that is, we have an expression which we cannot use arithmetically unless x has a properly chosen value. If we were studying some problem which involved such a series, it would be a matter of the most vital importance to know whether the values of x under discussion were such as to make the series meaningless.

This question of distinguishing between expressions which converge (that is, the sum of whose terms approaches a limit) and those which do not has an interesting history. Newton and his followers in the seventeenth century dealt with infinite series and always assumed that they converged, as, in fact, most of them did. But as more complicated series came into use it became more difficult to tell from inspection whether they meant anything or not for a given value of the variable.

It was not until the beginning of the nineteenth century that Gauss, Abel, and Cauchy, in Germany, Norway, and France respectively, began to study this subject effectively and to devise far-reaching tests to determine the values of x for which certain series converge to a finite limit. It is said that on hearing a discussion by Cauchy in regard to series which do not always converge, the astronomer La Place became greatly alarmed lest he had made use of some such series in his great work, "Celestial Mechanics." He hurried home and denied himself to all distractions until he had examined every series in his book. To his intense satisfaction they all converged. In fact it has often been observed that a genius can safely take chances in the use of delicate processes, which seem very foolish and unsafe to a man of ordinary ability.

MISCELLANEOUS EXERCISES

1. The digits of a certain three-digit number are in geometrical progression. The sum of the digits is 13. The number divided by the sum of its digits gives a quotient of 10 and a remainder of 9. Find the number.

2. What per cent of the eleventh term of the progression 1, 2, 4, 8, ⋯ is the eleventh term of 1, 101, 201, ⋯?

3. Compare the sum of the first ten terms of the first progression in Exercise 2 with the sum of the first ten terms of the second progression.

4. What meaning may be attached to (*a*) $\$500 \cdot (1.06)^3$? (*b*) $\$500 \cdot (1.05)^5$? (*c*) $\$500 \cdot (1.03)^6$?

5. What will \$100 amount to in three years, with interest at 4%, compounded annually? in five years?

6. What will \$100 amount to in two years, with interest at 6%, compounded semiannually? in three years?

7. For each of ten successive years a man saves $200 from his salary and places it in a savings bank where it draws 4% interest, compounded annually. Find the total amount of his savings at the time of his fifth annual deposit. Indicate (without multiplication) the amount of his deposit at the beginning of the eleventh year.

8. A loan of S dollars is to be repaid in four equal annual payments of p dollars each. Find p if money is worth $r\%$.

Solution. The sum due at beginning of second year is

$$S(1+r)-p. \tag{1}$$

The sum due at beginning of third year is

$$[S(1+r)-p](1+r)-p. \tag{2}$$

The sum due at beginning of fourth year is

$$\{[S(1+r)-p](1+r)-p\}(1+r)-p. \tag{3}$$

The sum due at beginning of fifth year is

$$[\{[S(1+r)-p](1+r)-p\}(1+r)-p](1+r)-p. \tag{4}$$

By the conditions of the problem, (4) = 0, for all the debt has then been paid. Setting (4) equal to zero and simplifying,

$$S(1+r)^4-p(1+r)^3-p(1+r)^2-p(1+r)-p=0. \tag{5}$$

Solving (5) for p,

$$p=\frac{S(1+r)^4}{(1+r)^3+(1+r)^2+(1+r)+1}. \tag{6}$$

But the denominator in (6) is a geometrical series whose sum by the formula $S_n=\frac{a-ar^n}{1-r}$ is $\frac{(1+r)^4-1}{r}$.

Substituting this last value for the denominator of (6),

$$p=\frac{Sr(1+r)^4}{(1+r)^4-1}. \tag{7}$$

In the general case, if we have n annual payments, the exponent 4 in (7) would be replaced by n, and then $p=\frac{Sr(1+r)^n}{(1+r)^n-1}$.

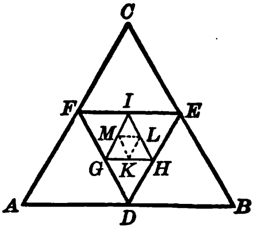
9. A loan of $1000 is to be repaid in three equal annual payments, interest at 6%. Find the payment required.

10. A loan of $2000 bearing interest at 5% is to be repaid in five equal annual payments. Find the payment required.

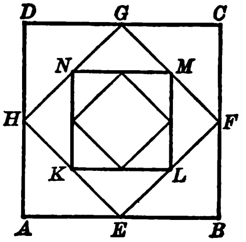
11. The machinery of a certain mill cost $10,000. The owner figures that the machinery depreciates 10% in value each successive year. What was the estimate on the value of the machinery at the end of the sixth year?

12. A vessel containing wine was emptied of one third of its contents and then filled with water. This was done four times. What portion of the original contents was then in the vessel?

13. In the adjacent figure the triangle DEF is formed by joining the mid-points of AB, BC, and CA respectively. Triangles GHI and KLM are formed in like manner. If ABC is equilateral, prove that the successive perimeters of the triangles form a geometrical progression. If $AB = 5$, find the sum of the perimeters of all the triangles which may be so formed.



14. Each square in the adjacent figure except the first is formed by joining the mid-points of the square next larger. If $AB = 4$, show that the perimeters of the squares form a decreasing geometrical progression. Find the sum of the perimeters of all the squares which may be so drawn.



CHAPTER XXXIII

THE BINOMIAL THEOREM

221. Powers of binomials. The following identities are easily obtained by actual multiplication:

$$(a+b)^2=a^2+2\,ab+b^2. \tag{1}$$

$$(a+b)^3=a^3+3\,a^2b+3\,ab^2+b^3. \tag{2}$$

$$(a+b)^4=a^4+4\,a^3b+6\,a^2b^2+4\,ab^3+b^4. \tag{3}$$

$$(a+b)^5=a^5+5\,a^4b+10\,a^3b^2+10\,a^2b^3+5\,ab^4+b^5. \tag{4}$$

If $a+b$ is replaced by $a-b$, the even-numbered terms in each of the preceding expressions will then be negative and the odd-numbered terms will be positive.

222. The expansion of $(a+b)^n$. The form of the expansion for the general case will now be indicated:

The first term is a^n *and the last is* b^n.

The second term is $+na^{n-1}b$.

The exponents of a *decrease by 1 in each term after the first.*

The exponents of b *increase by 1 in each term after the second.*

The product of the coefficient in any term and the exponent of a *in that term, divided by the exponent of* b *increased by 1, gives the coefficient of the next term.*

The sign of each term is $+$ *if* a *and* b *are positive; the sign of each even-numbered term is* $-$ *if* b *alone is negative.*

According to the above statement we have

$$(a+\mathrm{b})^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \ldots + b^n.$$

This expresses in symbols the law known as the **binomial theorem.** The theorem holds for all positive values of n and also, with certain limitations, for negative values.

NOTE. The coefficients of the various terms in the binomial expansion are displayed in a most elegant form as follows:

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

In this arrangement each row may be derived from the one above it by observing that each number is equal to the sum of the two numbers, one to the right and the other to the left of it, in the line above. Thus $4 = 1 + 3$, $6 = 3 + 3$, etc. The next line is 1 5 10 10 5 1. The successive lines of this table give the coefficients for the expansions of $(a+b)^n$ for the various values of n. The numbers in the last line of the triangle are seen to be the coefficients when $n = 4$; the next line would give those for $n = 5$. This array is known as Pascal's triangle, and was published in 1665. It was probably known to Tartaglia nearly a hundred years before its discovery by Pascal.

ORAL EXERCISES

What is the second term in the expansions of Exercises 1–4?

1. $(a+b)^{18}$. **2.** $(a-b)^{27}$. **3.** $(a+b)^{21}$. **4.** $(a-b)^{31}$.

Assuming that the terms in Exercises 5–10 occur in an expansion of the binomial $a+b$, find (a) the exponents, (b) the coefficient in the next following term of the expansion.

5. $10a^2b^3$.

6. $3ab^2$.

7. $15a^4b^2$.

8. $11\cdot 5\cdot 9a^8b^4$.

9. $252a^5b^5$.

10. $20a^3b^3$.

EXERCISES

Expand by the rule:

1. $(a+b)^7$.
2. $(a-1)^6$.
3. $(a+1)^7$.
4. $(a+2)^5$.
5. $(a+3)^6$.
6. $(2-a)^7$.

Obtain the first four terms of the following:

7. $(a+b)^{21}$. 8. $(a+b)^{30}$. 9. $(a+1)^{32}$. 10. $(a-2)^{25}$.

Expand:

11. $(a^2+2b)^5$.

Hints. To avoid confusion of exponents first write

$$(a^2)^5 + \quad (a^2)^4(2b)^1 + \quad (a^2)^3(2b)^2 + \quad (a^2)^2(2b)^3 + \quad (a^2)^1(2b)^4 + (2b)^5.$$

Then in the spaces left for them put in the coefficients according to the rule of this section.

Finally, expand and simplify each term.

12. $(a^2+2)^6$. 13. $(a^2-2b)^7$. 14. $\left(\frac{a^2+1}{b}\right)^5$. 15. $\left(\frac{a^2-1}{a^3}\right)^6$.

Obtain in simplest form the first four terms of the following:

16. $(a^2+3b)^{20}$.

17. $(a^2-3b^2)^{10}$.

18. $\left(\frac{a^2-1}{a}\right)^{30}$.

19. $\left(\frac{a}{b}+\frac{b}{a}\right)^{20}$.

20. $\left(\frac{2x}{y^3}-\frac{y^4}{x^5}\right)^8$.

21. $\left(\frac{a^2}{b^3}+\frac{2b^2}{a^4}\right)^{12}$.

22. $\left(\frac{3x^5}{y^3}-\frac{2y^{15}}{9x^{12}}\right)^5$.

23. $\left(\frac{\sqrt{x}}{y}+\frac{\sqrt{y}}{x}\right)^{12}$.

24. $\left(1+\frac{1}{n}\right)^n$.

25. Write the first six terms of the expansion of $(a+b)^n$, and evaluate it for $n=1$, $n=2$, $n=3$, $n=4$. How does the number of terms compare with n? What is the value of each coefficient after the $(n+1)$th? Why does not the expansion extend to more than five terms when $n=4$?

Compute the following to two decimal places. (In each case carry the computation far enough to be certain that the terms neglected do not affect the second decimal place.)

26. $(1.1)^{10}$.

HINT. $(1.1)^{10} = (1 + .1)^{10}$ etc.

27. $(5.2)^8$.

28. $(1.06)^6$.

29. $(.98)^{11}$.

HINT. $(.98)^{11} = (1 - .02)^{11}$ etc.

30. $(4.9)^9$.

31. $(2.9)^8$.

223. Extraction of roots by use of the binomial expansion. The expansion in section 222 may be verified for any particular integral value of n without difficulty by direct multiplication, as in section 221. But if n has a negative or fractional value, a laborious proof is required to show that the expansion is still valid when a is numerically greater than b. Since none of the factors of the coefficients, as n, $n-1$, $n-2$, vanish for fractional or negative values of n, it appears that for such exponents the expansion is an infinite series.

For example,

$$(a+b)^{\frac{1}{2}} = a^{\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{1}{2}}b - \tfrac{1}{8} \cdot a^{-\frac{3}{2}}b^2 + \tfrac{1}{16} \cdot a^{-\frac{5}{2}}b^3 + \cdots.$$

By giving n the values $\frac{1}{2}$ or $\frac{1}{3}$, one can compute the square root or the cube root of a number to any required degree of accuracy.

In such computations it is desirable to let the number which corresponds to a in the binomial exceed the one corresponding to b by as much as possible and at the same time to have $a^{\frac{1}{n}}$ an integer.

NOTE. The process of extracting the square root and even the cube root by means of the binomial expansion was familiar to the Hindus more than a thousand years ago. The German Stifel (1486–1567) stated the binomial theorem for all powers up to the seventeenth, and also extracted roots of numbers by this method.

EXERCISES

Find the first four terms of the following:

1. $(1+x)^{\frac{1}{2}}$. **3.** $(3-x)^{\frac{1}{2}}$. **5.** $(2+x)^{\frac{1}{3}}$.

2. $(2+x)^{\frac{1}{2}}$. **4.** $(1+x)^{\frac{1}{3}}$. **6.** $(3-x)^{\frac{1}{3}}$.

Find to at least three decimals by the binomial theorem:

7. $(27)^{\frac{1}{2}}$.

Solution.
$$\begin{aligned}(27)^{\frac{1}{2}} &= (25+2)^{\frac{1}{2}}\\ &= 25^{\frac{1}{2}} + \tfrac{1}{2}\cdot 25^{-\frac{1}{2}}\cdot 2 - \tfrac{1}{8}\cdot 25^{-\frac{3}{2}}\cdot 2^2\\ &\qquad + \tfrac{1}{16}\cdot 25^{-\frac{5}{2}}\cdot 2^3 - \cdots\\ &= 5 + .2 - .004 + .00016 \cdots = 5.196+.\end{aligned}$$

It is proved in more advanced books that when the terms of an infinite series are alternately plus and minus, and each term is numerically less than the preceding one, the value of the entire sum from a given term on cannot exceed that term. This fact renders these so-called "alternating series" especially convenient for computation, since a definite limit of error is known at each stage of the computation. In this example the error cannot exceed .00016.

8. $(17)^{\frac{1}{2}}$. **9.** $(28)^{\frac{1}{2}}$. **10.** $(38)^{\frac{1}{2}}$. **11.** $(78)^{\frac{1}{2}}$. **12.** $(125)^{\frac{1}{2}}$.

13. $(61)^{\frac{1}{3}}$.

Solution.
$$\begin{aligned}(61)^{\frac{1}{3}} &= (64-3)^{\frac{1}{3}}\\ &= 64^{\frac{1}{3}} - \tfrac{1}{3}\cdot 64^{-\frac{2}{3}}\cdot 3 - \tfrac{1}{9}\cdot 64^{-\frac{5}{3}}\cdot 3^2\\ &\qquad - \tfrac{5}{81}\cdot 64^{-\frac{8}{3}}\cdot 3^3 - \cdots\\ &= 4 - \tfrac{1}{16} - \tfrac{1}{1024} - \tfrac{5}{196608} - \cdots\\ &= 4 - .0625 - .00097 - .00000259 \cdots = 3.93653-.\end{aligned}$$

Here three terms give the result to five figures.

14. $(79)^{\frac{1}{2}}$.

HINT. $(79)^{\frac{1}{2}} = (81-2)^{\frac{1}{2}} = 81^{\frac{1}{2}} - \frac{1}{2}\cdot 81^{-\frac{1}{2}}\cdot 2 - \cdots$.

Here $(81-2)^{\frac{1}{2}}$ yields more accurate results with fewer terms than does $(64+15)^{\frac{1}{2}}$.

15. $(28)^{\frac{1}{3}}$. **16.** $(66)^{\frac{1}{3}}$. **17.** $(30)^{\frac{1}{3}}$. **18.** $(700)^{\frac{1}{3}}$.

Exponent of 2nd term is 1 less than no. of term

224. The factorial notation. The notation $5!$ or $\underline{|5}$ signifies $1\cdot 2\cdot 3\cdot 4\cdot 5$, or 120. Also $4! = 1\cdot 2\cdot 3\cdot 4 = 24$.

In general, $n! = 1\cdot 2\cdot 3\cdot 4\cdots(n-2)(n-1)n$.

The symbol $n!$ or $\underline{|n}$ is read "factorial n."

In the factorial notation the denominators of the fourth and fifth terms of the expansion of $(a+b)^n$ become $3!$ and $4!$ respectively (see formula, p. 456).

EXERCISES

Evaluate:

1. $6!$.
2. $4!\cdot 3!$.
3. $5!\cdot 2!$.
4. $6!\div 2!$.
5. $4!-3!\cdot 2!$.
6. $n!\div(n-1)!$.

Evaluate $\dfrac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}$, when:

7. $n=7, r=5$.
8. $n=15, r=8$.
9. $n=21, r=12$.
10. $n=20, r=15$.
11. $n=18, r=17$.
12. $n=10, r=11$.

225. The rth term of $(a+b)^n$. According to the binomial theorem the fifth term of the expansion on page 456 is

$$+\frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{4!}.$$

If we note carefully this term and the directions on page 455, we can write down, from the considerations that follow, any required term without writing other terms of the expansion.

The *denominator of the coefficient* of the fifth term is $4!$ From the law of formation the denominator of the sixth term would be $5!$, of the seventh term $6!$, etc. Consequently in the rth term the denominator of the coefficient would be $(r-1)!$.

The *numerator of the coefficient* of the fifth term contains the product of the four factors $n(n-1)(n-2)(n-3)$. The numerator of the sixth term would contain these four and the additional factor $n-4$. Similarly, the last factor in the numerator of the seventh term would be $n-5$, etc. Hence the last factor in the rth term would be $n-(r-2)$, and the numerator of the coefficient of the rth term is $n(n-1)(n-2)(n-3)\cdots(n-r+2)$.

The *exponent of a* in the fifth term is $n-4$, and in the sixth term it would be $n-5$, etc. Therefore in the rth term the exponent of a is $n-(r-1)$, or $n-r+1$.

The *exponent of b* in the fifth term is 4, in the sixth term is 5, etc. Therefore in the rth term the exponent of b is $r-1$.

The *sign* of any term of the expansion (if n is a positive integer) is plus if the binomial is $a+b$. If the binomial is $a-b$, the terms containing the odd powers of b will be negative. In other words, the sign in such cases depends upon whether the exponent $r-1$ is odd or even.

Hence the rth term (r not equal to 1) of $(a+b)_n$ equals

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}. \tag{1}$$

If we wanted the twelfth term, we would in using (1) substitute 12 for r.

EXERCISES

Write the indicated terms:

1. Fifth term of $(a+b)^{10}$.

Solution. Substituting 10 for n and 5 for r in the formula (1) gives

$$\frac{10\cdot9\cdot8\cdot7}{4!}a^6b^4=\frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2}a^6b^4$$
$$=210\,a^6b^4.$$

2. Sixth term of $(a+b)^9$.

3. Fourth term of $(a+b)^{20}$.

4. Seventh term of $(a-b)^{10}$.

5. Eighth term of $(a-b)^{15}$.

6. Fourth term of $\left(a+\frac{1}{a}\right)^{30}$. **8.** Sixth term of $\left(\frac{a}{b}-\frac{b^2}{a}\right)^{18}$.

7. Fifth term of $(a^2-b)^{20}$. **9.** Middle term of $(x^2-x)^{16}$.

10. Seventh term of $\left(\frac{a^2}{b}-\frac{2b^2}{a}\right)^{14}$.

11. Fifth term of $\left(\sqrt{x}-\sqrt{\frac{y}{x}}\right)^{15}$.

Find the coefficient of:

12. x^5 in $(1+x)^{10}$. **14.** x^{15} in $(x^3+1)^{15}$.

13. x^8 in $(1+x^2)^{16}$. **15.** x^{10} in $(x^3-x^{-1})^{14}$.

NOTE. The binomial theorem occupies a remarkable place in the history of mathematics. By means of it Napier was led to the discovery of logarithms, and its use was of the greatest assistance to Newton in making his most wonderful mathematical discoveries. But to-day the results of Newton and of Napier are explained without even so much as a mention of the binomial theorem, for simpler methods of obtaining these results have been discovered.

It was Newton who first recognized the truth of the theorem, not only for the case where n is a positive integer, which had long been familiar, but for fractional and negative values as well. He did not give a demonstration of the general validity of the binomial development, and none even passably satisfactory was given until that of Euler (1707–1783). The first entirely satisfactory proof of this difficult theorem was given by the brilliant young Norwegian Abel (1802–1829).

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JOHN NAPIER

CHAPTER XXXIV

RATIO, PROPORTION, AND VARIATION

226. Ratio. The **ratio** of one number a to a second number b is the quotient obtained by dividing the first by the second, or $\frac{a}{b}$. The ratio of a to b is also written $a:b$.

It follows from the above that all ratios of two numbers are fractions and all fractions may be regarded as ratios.

Thus $\frac{2}{5}$, $\frac{a}{3d}$, $\frac{a+2}{c-d}$, and $\frac{\sqrt{2}}{\sqrt{3}}$ are ratios as well as fractions.

Since ratios like the above are fractions, operations which may be performed on fractions may be performed on these ratios. Hence the value of a ratio is not changed by multiplying or dividing both numerator (antecedent) and denominator (consequent) by the same number.

Thus $$\frac{a}{b}=\frac{a\cdot x}{b\cdot x} \text{ and } \frac{a}{b}=\frac{\frac{a}{y}}{\frac{b}{y}}.$$

EXERCISES

Simplify the following ratios by considering them as fractions and reducing the fractions to lowest terms:

1. $\left(\frac{4a^2+a-3}{a^2-1}\right):\left(4+\frac{4a-1}{a^2-a-2}\right)$.

2. 1 kilometer : 1 mile. (1 kilometer = .62 miles.)

3. 1 liter : 1 quart. (1 liter = .001 cubic meter; 1 quart = $\frac{231}{4}$ cubic inches; and 1 meter = 39.4 inches.)

4. A city lot 100×160 feet : 1 acre. (1 acre = 43,560 square feet.)

5. Area of printed portion of this page : total area of the page.

6. If $24,000 is divided between two men so that the portions received are to each other as $5:7$, how much does each receive?

Hint. Let $5x$ and $7x$ be the required parts.

7. Separate 690 into four parts which are to each other as $2:5:7:9$.

8. Show that $\frac{x}{x+2} < \frac{x+3}{x+5}$ if x is positive.

Hint. Reduce the given fractions to respectively equivalent fractions having a common denominator, then compare the numerators of the fractions so obtained.

9. Arrange the ratios $3:4$ and $7:9$ in decreasing order of magnitude.

227. Proportion. A **proportion** is a statement of equality between two ratios. Four numbers, a, b, c, and d, are in proportion if the ratio of the first pair equals the ratio of the second pair.

This proportion is written

$$a:b=c:d \quad \text{or} \quad \frac{a}{b}=\frac{c}{d}.$$

Though both forms are equations, the second is the more familiar one and for this reason is preferable.

Note. By the earlier mathematicians ratios were not treated as if they were numbers, and the equality of two ratios which we know as a proportion was not denoted by the same symbol as other kinds of equality. The usual sign of equality for ratios was $::$, a notation which was introduced by the Englishman Oughtred in 1631 and was brought into common use by John Wallis about 1686. The sign = was used in this connection by Leibnitz (1646–1716) in Germany, and by the Continental writers generally, while the English clung to Oughtred's notation.

In the proportion $\frac{a}{b}=\frac{c}{d}$ the first and fourth terms (a, d) are called the **extremes**, and the second and third terms (b, c) are called the **means**.

228. Mean proportional. A **mean proportional** between two numbers a and b is the number m if $\frac{a}{m}=\frac{m}{b}$. It follows that $m^2=ab$, or $m=\pm\sqrt{ab}$.

229. Third proportional. A **third proportional** to two numbers a and b is the number t if $\frac{a}{b}=\frac{b}{t}$.

230. Fourth proportional. A **fourth proportional** to three numbers a, b, and c is the number f if $\frac{a}{b}=\frac{c}{f}$.

231. Test of a proportion. Since a proportion is an equality between two ratios (fractions), it is therefore an equation. Hence *any operation which may be performed on an equation may be performed on a proportion.* (See Axioms, pp. 39–40.)

Thus, in the proportion $\frac{a}{b}=\frac{c}{d}$ both members may be multiplied by bd, giving $ad = bc$. Here the first member is the product of the extremes of the proportion, and the second member is the product of the means.

Therefore *in any proportion the product of the extremes equals the product of the means.*

232. Proportions from equal products. The numbers which occur in a pair of equal products may be used in various ways as the terms of a proportion.

Thus, if $ad = bc$,

we may write either $\frac{a}{b} = \frac{c}{d}$, or $\frac{a}{c} = \frac{b}{d}$.

Proof. If $a \cdot d = b \cdot c$ is divided by bd, we obtain

$$\frac{ad}{bd} = \frac{bc}{bd}, \text{ or } \frac{a}{b} = \frac{c}{d}. \quad (1)$$

If $a \cdot d = b \cdot c$ is divided by cd, we obtain

$$\frac{a}{c} = \frac{b}{d}. \quad (2)$$

If the means in (1) are interchanged, (2) is obtained. This process of obtaining (2) from (1) is called **alternation.**

If $a \cdot d = b \cdot c$ is divided by ac, we obtain

$$\frac{b}{a} = \frac{d}{c}. \quad (3)$$

If the fractions in (1) are inverted, (3) is obtained. This process of obtaining (3) from (1) is called **inversion.**

EXERCISES

1. Find a mean proportional between: (*a*) 3 and 27; (*b*) $\frac{4}{7}$ and $\frac{7}{9}$; (*c*) $\frac{81}{28}$ and $\frac{7}{9}$; (*d*) $\frac{x}{y}$ and xy.

2. Find a third proportional to: (*a*) 9 and 6; (*b*) 180 and 60; (*c*) 216 and 36.

3. Find the fourth proportional to: (*a*) 14, 10, and 7; (*b*) 27, 3, and 36; (*c*) 96, 12, and 8.

4. Form three proportions from each of the following equations: (*a*) $5x = 9y$; (*b*) $(a + 3) \cdot 2 = (a + 1) \cdot 3$; (*c*) $x^2 - y^2 = r^2 - s^2$.

5. Write by alternation: (*a*) $\frac{2}{3} = \frac{10}{15}$; (*b*) $\frac{2}{9} = \frac{20}{x}$.

6. Write by inversion: (*a*) $\frac{5}{6} = \frac{10}{12}$; (*b*) $\frac{2}{x} = \frac{12}{30}$.

If $\frac{a}{b} = \frac{c}{d}$, prove the following and state the corresponding theorems in words:

7. $\frac{a}{c} = \frac{b}{d}$.

8. $\frac{b}{a} = \frac{d}{c}$.

9. $\frac{a^n}{b^n} = \frac{c^n}{d^n}$.

10. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{\sqrt[n]{c}}{\sqrt[n]{d}}$.

11. $\frac{a+b}{b} = \frac{c+d}{d}$.

Hint. Add 1 to each member of $\frac{a}{b} = \frac{c}{d}$.

12. $\frac{a+b}{a} = \frac{c+d}{c}$.

Hint. Write $\frac{a}{b} = \frac{c}{d}$ by inversion and apply hint of Exercise 11.

13. $\frac{a-b}{b} = \frac{c-d}{d}$.

14. $\frac{a-b}{a} = \frac{c-d}{c}$.

15. $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

The proportions given in Exercises 11, 13, and 15 are said to be derived from $a : b = c : d$ by **addition**, **subtraction**, and **addition and subtraction** respectively.

If $\frac{a}{b} = \frac{c}{d}$, show that the following equalities are true:

16. $\frac{5a}{b} = \frac{5c}{d}$.

17. $\frac{2a}{7b} = \frac{2c}{7d}$.

18. $\frac{a^2}{b^2} = \frac{ac}{bd}$.

19. $\frac{5a+b}{5a-b} = \frac{5c+d}{5c-d}$.

20. $\frac{a^2-b^2}{b^2} = \frac{c^2-d^2}{d^2}$.

21. $\frac{a^2-7b^2}{b^2} = \frac{c^2-7d^2}{d^2}$.

22. $\frac{a^2-b^2}{2ab} = \frac{c^2-d^2}{2cd}$.

23. $\frac{7a^2-3b^2}{5ab} = \frac{7c^2-3d^2}{5cd}$.

24. $\frac{a^2+ab+b^2}{c^2+cd+d^2} = \frac{a^2-ab+b^2}{c^2-cd+d^2}$.

25. In the proof which follows give the reason for each step and state the result as a theorem:

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$.

Proof. Setting each of the given ratios above equal to r,

$$\frac{a}{b}=r,\quad \frac{c}{d}=r,\quad \text{and}\quad \frac{e}{f}=r. \qquad (1)$$

Then from (1), $a=br,\quad c=dr,\quad e=fr.$ (2)

Adding in (2), $a+c+e=br+dr+fr.$ (3)

Factoring in (3), $a+c+e=(b+d+f)r.$ (4)

Therefore $\frac{a+c+e}{b+d+f}=r.$ (5)

Hence by (1) and (5), $\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}.$

26. If $\frac{a}{b}=\frac{c}{d}=\frac{5}{6}$, show that $\frac{a+5}{b+6}=\frac{c}{d}$, or $\frac{5}{6}$.

Solve, using theorems of proportion:

27. $2x:(x+8)=10:3.$

28. $25:x=x:169.$

29. $5:4=(x-3):(x-4).$

30. $(15+x):(15-x)=13:17.$

31. $(10+x):(20+3x)=(10-x):-3x.$

32. $\frac{\sqrt{x}+2}{\sqrt{x}-2}=\frac{x+1}{x-7}.$

33. Show that the mean proportional between two numbers is the geometric mean between these numbers.

PROBLEMS

1. The surface of a sphere is $4\pi R^2$. If S represents the surface of a sphere, R its radius, and D its diameter, show that: (*a*) $\frac{S_1}{S_2}=\frac{R_1^2}{R_2^2}$; (*b*) $\frac{S_1}{S_2}=\frac{D_1^2}{D_2^2}$.

2. Find the ratio of the surfaces of two spheres whose radii are in the ratio 1:10.

3. If the diameter of the earth is 7920 miles and that of Mars is 4230 miles, find the ratio of their surfaces.

4. If the diameter of the moon is 2160 miles, find the ratio of its surface to that of the earth.

5. The volume of a sphere is $\frac{4\pi R^3}{3}$. If V represents the volume of a sphere, R its radius, and D its diameter, show that for any two spheres,

$$\frac{V_1}{V_2}=\frac{R_1^3}{R_2^3}=\frac{D_1^3}{D_2^3}.$$

6. The diameter of the sun is approximately one hundred and nine times the diameter of the earth. Find the ratio of their volumes.

7. From Exercises 3 and 5 find the ratio of the volumes of Mars and the earth.

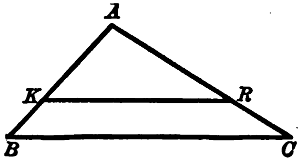
8. Find the ratio of the volumes of the earth and the moon.

9. The areas of two similar triangles are to each other as the squares of any two corresponding lines. If the corresponding sides of two similar triangles are 12 and 20 and the area of the first is 90 square inches, find the area of the second.

10. The areas of two similar triangles are 147 and 300 respectively. If the base of the first is 10, find the corresponding base of the second.

11. If ABC is any triangle and KR is a line parallel to BC, meeting AB at K and AC at R, then

$$\frac{\text{area } ABC}{\text{area } AKR}=\frac{AB^2}{AK^2}=\frac{AC^2}{AR^2}=\frac{BC^2}{KR^2}.$$



If in the accompanying figure area $ABC = 225$ square inches, area $AKR = 81$ square inches, and $AB = 15$ inches, find AK.

12. In the figure of Exercise 11, if $ABC = 845$ square inches, $BC = 13$ inches, and $KR = 7$ inches, find the area AKR.

13. If in the figure of Exercise 11 triangle AKR equals $\frac{4}{21}$ of the trapezoid $KBCR$ and $AC = 15$ inches, find AR and RC.

14. In Exercise 13 substitute $\frac{1}{5}$ for $\frac{4}{21}$ and solve for AR to two decimals.

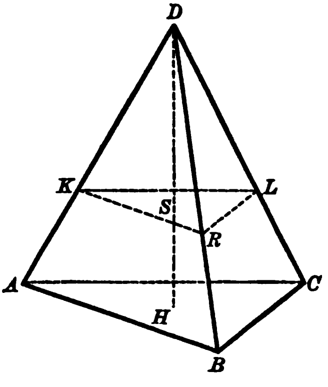
15. If in the figure of Exercise 11 the triangle is equivalent to the trapezoid and $AK = 10$, find KB to two decimals.

16. A certain flagpole casts a shadow 45 feet long at the same time that a near-by post 8 feet high casts a shadow $4\frac{1}{2}$ feet long. Find the height of the pole.

17. The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides. In triangle ABC if $AB = 15$, $BC = 24$, and $CA = 25$, find the segments of BC made by the bisector of angle A.

18. If a plane be passed parallel to the base of a pyramid (or cone), as in the accompanying figure, cutting it in KRL, then pyramid $D - ABC$: pyramid $D - KRL = DH^3 : DS^3$, etc.

If in the adjacent figure the volumes of the pyramids are 8 and 27 cubic inches respectively, and the altitude DH equals 15 inches, find DS.



19. If DH in the accompanying figure is 18 inches and the volume of one pyramid is one third the volume of the other, find DS to two decimals.

20. In the accompanying figure the frustum is seven eighths of the whole pyramid. (*a*) If DH equals 16, find DS; (*b*) if $DH = x$, find DS.

21. If a plane parallel to the base divides the whole pyramid into two parts having equal volumes and $DH = 75$, find to two decimals the parts into which the plane divides DH.

22. The volumes of two similar figures are to each other as the cubes of any two corresponding edges. Compare the volumes of two similar solids, the edge of one of which is 50% greater than the corresponding edge of the other.

23. Compare the radii of two spheres whose volumes are to each other as $125 : 27$.

233. Variation. The word **quantity** denotes anything which is **measurable**, such as distance, rate, time, and area.

Many operations and problems in mathematics deal with numerical measures of quantities, some of which are fixed and others constantly changing. Such problems as deal with the relation of the numerical measures of at least two changing quantities are called problems in variation

The theory of variation is really involved in proportion. This fact will become evident after a study of the illustrations of the different kinds of variation here given.

The equation $x = 3y$ may refer to no physical quantities whatever, yet it is possible to imagine y as taking on in succession every possible numerical value, and the value of x as changing with every change of y, and consequently always being three times as great as the corresponding value of y. In this sense, which is strictly mathematical, x and y are variables.

The symbol for variation is $\propto$, and $x \propto y$ is read "x varies directly as y" or "x varies as y."

234. Direct variation. One hundred feet of copper wire of a certain size weighs 32 pounds. Obviously a piece of the same kind 200 feet long would weigh 64 pounds, a piece 300 feet long would weigh 96 pounds, and so on.

Here we have two variables, W (weight) and L (length), so related that the value of W depends on the value of L, and in such a way that W increases proportionately as L increases. That is, W is directly proportional, or merely proportional, to L. Hence, if W_1 and W_2 are any two weights corresponding to the lengths L_1 and L_2 respectively,

$$W_1 : W_2 = L_1 : L_2. \tag{1}$$

The fact expressed by (1) can be stated in the form of a variation, thus: $W \propto L$.

In general, if $x \propto y$, and x and y denote *any* two corresponding values of the variables, and x_1 and y_1 a *particular* pair of corresponding values of these variables,

then
$$\frac{x}{x_1} = \frac{y}{y_1}. \tag{2}$$

From (2),
$$x = \left(\frac{x_1}{y_1}\right) y. \tag{3}$$

But $\frac{x_1}{y_1}$ is a constant, being the quotient of two definite numbers.

Call this constant K, and (3) may be written

$$x = Ky.$$

That is, *if one variable varies as a second, the first always equals the second multiplied by some constant.*

Thus for the copper wire just mentioned, $W = \frac{32}{100} L$, or $\frac{8}{25} L$. Here, though W varies as L varies, W is always equal to L multiplied by the constant $\frac{8}{25}$.

EXERCISES

1. If $x \propto y$, and $x = 3$ when $y = 8$, find x when $y = 12$.

Solution. The variation is direct.

Therefore
$$\frac{3}{x} = \frac{8}{12}.$$

Solving,
$$x = 4\tfrac{1}{2}.$$

2. If $x \propto y$, and $x = 8$ when $y = 15$, find y when $x = 10$.

3. If $x \propto y$, and $x = h$ when $y = k$, find y when $x = r$.

4. If $x \propto y$, and $x = 2$ when $y = 5$, find K.

235. Inverse variation. If a tank full of water is emptied in 24 minutes through a smooth outlet in which the area of the opening A is 1 square inch, an outlet in which A is 2 square inches would empty the tank in one half the time, or in 12 minutes; and an outlet in which A is 3 square inches would empty the tank in 8 minutes.

Suppose it possible to increase or decrease A at will. When A is doubled t is halved; when A is trebled t is divided by 3; and so on. We then have in t (the time required to empty the tank) and in A (the area of the opening) two related variables such that if A increases, t will decrease proportionally, while if A decreases, t will increase proportionally.

Now let t_1 and t_2 be *any two* times corresponding to the areas A_1 and A_2 respectively; then

$$t_1 : t_2 = A_2 : A_1. \qquad (1)$$

The letters and the subscripts in (1) say: *The first time is to the second time as the second area is to the first area.*

The proportion (1) may be written $t_1 : t_2 = \frac{1}{A_1} : \frac{1}{A_2}$ where the subscripts on the t's and those on the A's come in the same order.

First, from (1), $$t_1 \cdot A_1 = t_2 \cdot A_2. \qquad (2)$$

Dividing (2) by A_1A_2, $$\frac{t_1}{A_2} = \frac{t_2}{A_1}. \qquad (3)$$

Whence $$t_1\left(\frac{1}{A_2}\right) = t_2\left(\frac{1}{A_1}\right). \qquad (4)$$

Therefore $$t_1 : t_2 = \frac{1}{A_1} : \frac{1}{A_2}. \qquad (5)$$

In the form of a variation (5) becomes $t \propto \frac{1}{A}$.

In general, x varies **inversely** as y when x varies as the **reciprocal** of y; that is,

$$x \propto \frac{1}{y}. \qquad (6)$$

And if x and y denote any two corresponding values of the variable, and x_1 and y_1 a particular pair of corresponding values,

$$x : x_1 = \frac{1}{y} : \frac{1}{y_1}. \qquad (7)$$

Whence $$\frac{x}{y_1} = \frac{x_1}{y}, \text{ or } xy = x_1y_1. \qquad (8)$$

But x_1y_1 is a constant, being the product of two definite numbers. Call this constant K.

Then (8) becomes $xy = K$.

That is, *if one variable varies inversely as another, the product of the two is a constant.*

EXERCISES

1. If x varies inversely as y, and $x = 8$ when $y = 5$, find x when $y = 15$.

Solution. The variation is inverse.

Hence $$8 : x = \frac{1}{5} : \frac{1}{15}.$$

Solving, $x = 2\frac{2}{3}$.

2. If $x \propto \frac{1}{y}$, and $x = 1$ when $y = 25$, find x when $y = 10$.

3. If $y \propto \frac{1}{z}$, and $y = h$ when $z = k$, find y when $z = r$.

4. If $m \propto \frac{1}{n}$, and $m = 2$ when $n = \frac{1}{3}$, find m when $n = 12$.

5. If $t \propto \frac{1}{n}$, and $t = 2$ when $n = 8$, find n when $t = 8$.

6. If $w \propto \frac{1}{d}$, and $w = 100$ when $d = 4000$, find w when $d = 5000$.

7. If $t \propto \frac{1}{r}$, and $t = 4$ when $r = 25$, find K.

8. If $w \propto \frac{1}{d}$, and $w = 200$ when $d = 4000$, find K.

236. Joint variation. If the base of a triangle remains constant while the altitude varies, the area will vary as the altitude. Similarly, if the base varies while the altitude remains constant, the area will vary as the base. If both base and altitude vary, the area varies as the product of the two; that is, the area of the triangle varies **jointly** as the base and altitude. Further, if at any time A denotes the area of a variable triangle, and h_1 and b_1 the corresponding altitude and base, then

$$A_1 = \frac{h_1 b_1}{2}. \tag{1}$$

If A_2 denotes the area at *any other* time, and h_2 and b_2 the corresponding altitude and base, then

$$A_2 = \frac{h_2 b_2}{2}. \tag{2}$$

Now (1) ÷ (2) gives $A_1 : A_2 = h_1 b_1 : h_2 b_2$.

In the form of a variation this last proportion becomes

$$A \propto hb.$$

In general, any variable x varies jointly as two others, y and z, if

$$x \propto yz; \tag{1}$$

that is, if x varies as the product of the two.

If x varies jointly as y and z, and if x, y, and z denote *any* corresponding values of the variables, while x_1, y_1, and z_1 denote a *particular* set of such values, then

$$\frac{x}{x_1}=\frac{yz}{y_1z_1}. \tag{2}$$

From (2),

$$x=\left(\frac{x_1}{y_1z_1}\right)yz. \tag{3}$$

But in (3) the fraction $\frac{x_1}{y_1z_1}$ is a constant, since x_1, y_1, and z_1 are particular values of the variables x, y, and z. Calling this constant K, we may write $x \propto yz$ as the equation

$$x=Kyz.$$

One variable may vary directly as one variable (or several variables) and inversely as another (or several others). Also one variable may vary as the square, or the cube, or the square root, or the reciprocal, or as any algebraic expression whatever involving the other variable (or variables).

EXERCISES

1. If x varies jointly as y and z, and $x = 24$ when $y = 6$ and $z = 8$, find x when $y = 18$ and $z = 4$.

Solution. The variation is joint.

Therefore $$\frac{24}{x}=\frac{6\cdot 8}{18\cdot 4}.$$

Solving, $$x = 36.$$

2. If x varies as yz, and $x = 10$ when $y = 15$ and $z = 6$, find x when $y = 9$ and $z = 8$.

3. If $A \propto hb$, and $A = 30$ when $h = 5$ and $b = 12$, find A when $h = 7$ and $b = 10$.

4. If $A \propto hb$, and $A = 48$ when $h = 8$ and $b = 12$, find K.

5. If x varies directly as y and inversely as z, and $x = 10$ when $y = 5$ and $z = 27$, find x when $y = 12$ and $z = 36$.

6. If V varies directly as T and inversely as P, and $V = 80$ when $P = 30$ and $T = 300$, find P when $T = 400$ and $V = 40$.

PROBLEMS

1. The weight of any object below the surface of the earth varies directly as its distance from the center. An object weighs 172 pounds at the surface of the earth. What would be its weight (*a*) 1000 miles below the surface? (*b*) 3000 miles below the surface? (*c*) at the center of the earth? (Radius of the earth = 4000 miles.)

2. The distance which sound travels varies directly as the time. A soldier measures with a stop watch the time elapsing between the sight of the flash of an enemy's gun and the sound of its report. If sound travels 1100 feet per second, how far off was the enemy when the observed time was $5\frac{2}{5}$ seconds?

3. When the volume of air in a bicycle pump is 24 cubic inches, the pressure on the handle is 30 pounds. Later when the volume of air is 20 cubic inches, the pressure is 36 pounds. Assume that a proportion exists here, determine whether it is direct or inverse, and find the volume of the air when the pressure is 42 pounds.

4. The distance (in feet) through which a body falls from rest varies as the square of the time in seconds. If a body falls 16 feet in 1 second, how far will it fall in (*a*) 3 seconds? (*b*) 10 seconds?

5. The intensity (brightness) of light varies inversely as the square of the distance from the source of the light. A reader holds his book 3 feet from a lamp and later 6 feet distant. At which distance does the page appear the brighter? How many times as bright?

6. A lamp shines on the page of a book 5 feet distant. Where must the book be held so that the page will receive twice as much light? four times as much light?

7. The area of a circle varies as the square of its radius. The area of a certain circle is 154 square inches and its radius is 7 inches. Find the radius of a circle whose area is 616 square inches.

8. The area illuminated on a screen by a spot light varies directly as the square of the distance from the source of the light to the screen. If the lighted area at a distance of 40 feet is a circle of diameter 10 feet, find the diameter of the illuminated circle at a distance of 15 feet.

9. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. An object weighs 172 pounds at the surface of the earth. What would it weigh (*a*) 1000 miles above the surface? (*b*) 3000 miles above the surface? (*c*) 5000 miles above the surface?

10. How far above the surface of the earth would a 150-pound object have to be placed so that its weight would be reduced one third?

11. The weight of a sphere of given material varies directly as the cube of its radius. Two spheres of the same material have radii 3 inches and 5 inches respectively. The first weighs 8 pounds. Find the weight of the second.

12. The time required by a pendulum to make one vibration varies directly as the square root of its length. If a pendulum 100 centimeters long vibrates once in 1 second, find the time of one vibration of a pendulum 81 centimeters long.

13. Find the length of a pendulum which vibrates once in 2 seconds; once in 7 seconds.

14. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. The pressure on 1 square foot is .9 pound when the rate of the wind is 15 miles per hour. Find the velocity of the wind when the pressure on 1 square yard is 72.9 pounds?

15. The pressure of water on the bottom of a containing vessel varies jointly with the area of the bottom and the depth of the water. When the water is 1 foot deep the pressure on 1 square foot of the bottom is 62.5 pounds. Find the pressure on the bottom of a circular tank of 14 feet diameter in which the water is 10 feet deep.

16. The cost of ties for a railroad varies directly as the length of the road and inversely as the distance between the ties. The cost of ties for a certain piece of road, the ties being 2 feet apart, was $1320. Find the cost of ties for a piece twenty times as long as the first if the ties are $2\frac{1}{4}$ feet apart.

CHAPTER XXXV

LOGARITHMS

237. Introduction. **Logarithms** were invented to shorten the work of extended numerical computations which involve one or more of the operations of multiplication, division, involution, and evolution. Their use has decreased the labor of computing to such an extent that many calculations which would require hours without the use of logarithms can be performed with their aid in one tenth of the time or less.

238. Definition of logarithm and base. If we write the equation

$$n = b^l, \tag{1}$$

we express therein the essential relation between a **number**, n, and its **logarithm**, l, for a given **base**, b. In the notation of logarithms this is written

$$\log_b n = l, \tag{2}$$

and it is read "the logarithm of n to the base b equals l."

We can define verbally in one statement both logarithm and base as follows:

The logarithm of a given number is the exponent in the power to which another number, called the ***base****, must be raised in order to equal the given number.*

It is important to realize that equations (1) and (2) are merely two different ways of expressing precisely the same relations, one the exponential way, the other the logarithmic.

Above all it is necessary to keep in mind the fact that *a logarithm is an exponent.*

Thus in $32 = 2^5$ the given number is 32, the base is 2, and the logarithm is 5; that is, $\log_2 32 = 5$.

239. Systems of logarithms. The base of the *common*, or *Briggs*, system of logarithms is 10. Hence a table of common logarithms is really a table of exponents of the number 10. Since the greater portion of these exponents are approximate values of irrational numbers, it follows that computations by means of logarithms give only approximate results. Tables exist, however, in which each logarithm is given to twenty or more decimals; hence practically any desired degree of accuracy can be obtained by using the proper table. This system is used in numerical work almost exclusively. The table on pages 494–495 is a table of common logarithms carried to four decimal places.

The only other system of logarithms used in computations is called the *natural system.* It has for its base the irrational number 2.7182+, which is usually denoted by the letter e and is used mainly for theoretical purposes.

It can be proved that the laws given on pages 329–330, governing the use of rational exponents, hold for irrational exponents. In the work on logarithms this fact will be assumed.

ORAL EXERCISES

1. If $8 = 2^x$, $x = ?$ $\log_2 8 = ?$

2. If $1000 = 10^x$, $x = ?$ $\log_{10} 1000 = ?$

3. $\log_2 16 = ?$ $\log_3 81 = ?$ $\log_5 625 = ?$

4. If $b^3 = 64$, $b = ?$ If $b^5 = 243$, $b = ?$

5. $10^1 = ?$ $\log_{10} 10 = ?$

6. $10^2 = ?$ $\log_{10} 100 = ?$

7. $10^0 = ?$ $\log_{10} 1 = ?$

8. $3^0 = ?$ $\log_3 1 = ?$

9. $5^0 = ?$ $\log_5 1 = ?$

10. $n^0 = ?$ $\log_n 1 = ?$

	NUMBER	BASE	LOGARITHM		NUMBER	BASE	LOGARITHM
11.	8	2	?	**24.**	?	4	3
12.	32	2	?	**25.**	?	6	2
13.	64	4	?	**26.**	9	?	$\frac{1}{2}$
14.	125	5	?	**27.**	8	?	$\frac{3}{2}$
15.	1000	10	?	**28.**	16	?	$\frac{3}{4}$
16.	16	?	2	**29.**	2	4	?
17.	81	?	2	**30.**	3	9	?
18.	81	?	4	**31.**	2	8	?
19.	49	?	2	**32.**	3	81	?
20.	216	?	3	**33.**	7	49	?
21.	?	10	3	**34.**	8	4	?
22.	?	10	2	**35.**	4	8	?
23.	?	10	1	**36.**	.1	10	?

Read in the notation of logarithms:

37. $300 = 10^{2.48}$.

38. $65 = 10^{1.81}$.

39. $4 = 10^{.60}$.

40. $1 = 10^0$.

41. $.10 = 10^{-1}$.

42. $1730 = 10^{3.238}$.

43. $173 = 10^{2.238}$.

44. $1.73 = 10^{.238}$.

45. $.173 = 10^{-1+.238}$.

46. $.0173 = 10^{-2+.238}$.

Read Exercises 47–49 and 54–56 as powers of 10:

47. $\log 3 = .48$. **48.** $\log 20 = 1.301$. **49.** $\log 4.9 = .69$.

50. $\log_{10}100 + \log_{10}1000 + \log_{10}10{,}000 = ?$

51. $\log_{10}10 + \log_{10}.01 - \log_{10}1 = ?$

52. $\log_2 8 + \log_3 27 + \log_4 1 = ?$

53. $\log_3 9 + \log_4 64 = ?$

54. $\log 490 = 2.69$.

55. $\log .0049 = -3 + .69$.

56. $\log 381 = 2.58$.

Biographical Note. *John Napier.* Although many scientists have been honored with titles on account of their discoveries, very few of the titled aristocracy have become distinguished for their mathematical achievements. A notable exception to this rule is found in John Napier, Lord of Merchiston (1550–1617), who devoted most of his life to the problem of simplifying arithmetical operations. Napier was a man of wide intellectual interests and great activity. In connection with the management of his estate he applied himself most seriously to the study of agriculture, and experimented with various kinds of fertilizers in a somewhat scientific manner, in order to find the most effective means of reclaiming soil. He spent several years in theological writing. When the danger of an invasion by Philip of Spain was imminent he invented several devices of war. Among these were powerful burning mirrors, and a sort of round musket-proof chariot, the motion of which was controlled by those within, and from which guns could be discharged through little portholes.

But by far the most serious activity of Napier's life was the effort to shorten the more tedious arithmetical processes. He invented the first approximation to a computing machine, and also devised a set of rods, often called Napier's bones, which were of assistance in multiplication. His crowning achievement, however, was the invention of logarithms, to which he devoted fully twenty years of his life.

240. Steps preceding computation. Before computation by means of the table can be taken up, two processes requiring considerable explanation and practice must be mastered.

I. To find from the table the logarithm of a given number.

II. To find from the table the number corresponding to a given logarithm.

241. Characteristic and mantissa. Unless a number is an exact power of 10, its logarithm consists of an *integer* and a *decimal.*

This fact is illustrated in Exercises 37–46, p. 482.

The integral part of a logarithm is called its **characteristic.**

The decimal part of a logarithm is called its **mantissa.**

Log $200 = 2.301$. Here 2 is the characteristic and .301 is the mantissa.

The characteristic of any number is obtained not from a table of logarithms but by an inspection of the number itself, according to rules which will now be derived.

$10^4 = 10,000$; that is, the $\log 10,000 = 4$.
$10^3 = 1000$; that is, the $\log 1000 = 3$.
$10^2 = 100$; that is, the $\log 100 = 2$.
$10^1 = 10$; that is, the $\log 10 = 1$.
$10^0 = 1$; that is, the $\log 1 = 0$.
$10^{-1} = .1$; that is, the $\log .1 = -1$.
$10^{-2} = .01$; that is, the $\log .01 = -2$.
$10^{-3} = .001$; that is, the $\log .001 = -3$.

The preceding table indicates between what two integers the logarithm of a number less than 10,000 lies. This determines the characteristic.

Since 542 lies between 100 and 1000 (that is, between 10^2 and 10^3), log 542 must lie between 2 and 3 and must equal 2 (characteristic) plus a decimal (mantissa).

And since .0045 lies between .001 and .01 (that is, between 10^{-3} and 10^{-2}), $\log .0045 = -3$ plus a positive decimal or -2 plus a negative decimal.

For the determination of the characteristic of a positive number we have the following rules:

I. The characteristic of a number greater than 1 is one less than the number of digits to the left of the decimal point.

II. The characteristic of a number less than 1 is negative and numerically one greater than the number of zeros between the decimal point and the first significant figure.

Accordingly the characteristic of 2536 is 3; of 6 is 0; of .4 is -1; of .032 is -2; of .00036 is -4.

ORAL EXERCISES

What is the characteristic of the following:

1. 347.
2. 9864.
3. 95.
4. 7.
5. 35.
6. 972.
7. 9720.
8. 97200.
9. 97.2.
10. 9.72.
11. .972.
12. .0972.
13. .00972.
14. 30.467.
15. .5000.
16. .000375.

The table on pages 494–495 gives the mantissas of numbers from 10 to 999. Before each mantissa a decimal point is understood.

The numbers 5420, 542, 5.42, .0542, and .000542 are spoken of as composed of the same significant digits in the same order. They differ only in the position of the decimal point, and consequently their logarithms *to the base 10* will have different characteristics, but they will have the same mantissa.

The last two points are easily illustrated by any two numbers which have the same significant digits in the same order.

$$\log 5.42 = .734, \text{ or } 5.42 = 10^{.734};$$

$$5.42 \cdot 10^2 = 542 = 10^{.734} \cdot 10^2 = 10^{2.734}.$$

Therefore $\log 542 = 2.734$.

The property just explained does not belong to a system of logarithms in which the base is any number other than 10. Thus, if the base is 100, the most convenient number after 10, the logarithms of 5420, 542, 54.2, and 5.42 are respectively 1.8670, 1.3670, .8670, and .3670. While a certain regularity in characteristic and mantissa can be seen here, it is obvious that the rules for obtaining them would not be so simple as they are for the base 10. Moreover, it can be seen that tables of a given accuracy are far shorter when the base is 10 than they would be with any other base.

242. Use of the table. To obtain the logarithm of a number of three or fewer significant figures from the table, we have the

Rule. *Determine the characteristic by inspection.*

Find in column N *the first two significant figures of the given number. In the row with these and in the column headed by the third figure of the number, find the required mantissa.*

ORAL EXERCISES

Find the logarithm of the following:

1. 263. **4.** 56. **7.** 3.7. **10.** 7.
2. 375. **5.** 560. **8.** 3700. **11.** 932.
3. 729. **6.** 37. **9.** 5. **12.** .932.

Solution. The characteristic of .932 is -1 and the mantissa is .9694. Hence log $.932 = -1 + .9694$. This is usually written in the abbreviated form, $\bar{1}.9694$. The mantissa is always kept positive in order to avoid the addition and subtraction of both positive and negative decimals, which in ordinary practice contain from three to five figures. Negative characteristics, being integers, are comparatively easy to take care of. (The student should note that log .932 is really negative, being $-1 + .9694$, or $-.0306$.)

13. .563. **15.** .00376. **17.** .0202. **19.** 3.86.
14. .0637. **16.** .00468. **18.** 725000. **20.** .987.

243. Interpolation. The process of finding the logarithm of a number not found in the table, from the logarithms of two numbers which are found there, or the reverse of this process, is called **interpolation.**

If we desire the logarithm of a number not in the table, say 7635, we know that it lies between the logarithms of 7630 and 7640, which are given in the table. Since 7635 is halfway between 7630 and 7640, we assume, though it is not *strictly* true, that the required logarithm is halfway between their logarithms, 3.8825 and 3.8831. In order to find log 7635 we first look up log 7630 and log 7640 and then take half (or .5) their difference (this difference may usually be taken from the column headed D) and add it to log 7630. This gives

$$\log 7635 = 3.8825 + .5 \times .0006 = 3.8828.$$

Were we finding log 7638, we should take .8 of the difference between log 7630 and log 7640 and add it to log 7630 as follows:

$$\begin{aligned}\log 7638 &= 3.8825 + .8 \times .0006\\ &= 3.8825 + .00048\\ &= 3.8825 + .0005\\ &= 3.8830.\end{aligned}$$

Observe that in using four-place tables one should not carry results to five figures. If the fifth figure is 5, 6, 7, 8, or 9, omit it and increase the fourth figure by 1; that is, *obtain results to the nearest figure in the fourth place.*

For finding the logarithm of a number we have the

Rule. *Prefix the proper characteristic to the mantissa of the first three significant figures.*

Then multiply the difference between this mantissa and the next greater mantissa in the table (called the tabular difference) by the remaining figures of the number preceded by a decimal point.

Add the product to the logarithm of the first three figures, taking the nearest decimal in the fourth place.

In this method of interpolation we have assumed that the increase in the logarithm is directly proportional to the increase in the number. As has been said, this is not strictly true, yet the results here obtained are nearly always correct to the fourth decimal place.

EXERCISES

Find the logarithm of the following:

1. 3625.
2. 464.7.
3. 52.73.
4. 42.75.
5. 646.8.
6. 82.543.
7. 10.101.
8. 500.35.
9. 705.50.
10. 3.0075.
11. .00286.
12. .0007777.
13. 30.07.
14. 3.1416.
15. 2.71828.
16. .0023456.

244. Antilogarithms. An **antilogarithm** is the number corresponding to a given logarithm. Thus antilog 2 equals 100.

If we desire the antilogarithm of a given logarithm, say 4.7308, we proceed as follows: The mantissa .7308 is found in the *row* which has 53 in column N and in the *column* which has 8 at the top. Hence the first three significant figures of the antilogarithm are 538. Since the characteristic is 4, the number must have five digits to the left of the decimal point. Thus antilog $4.7308 = 53,800$.

Therefore, if the mantissa of a given logarithm is found in the table, its antilogarithm is obtained by the

Rule. *Find the row and the column in which the given mantissa lies. In the row found take the two figures which are in column* N *for the first two significant figures of the antilogarithm and for the third figure the number at the top of the column in which the mantissa stands.*

Place the decimal point as indicated by the characteristic.

ORAL EXERCISES

Find the antilogarithm of the following:

1. 3.8768.
2. 1.8035.
3. .5763.
4. $\overline{1}.3747$.
5. $\overline{2}.7649$.
6. $7.5866 - 10$.

HINT. $7.5866 - 10 = \overline{3}.5866$.

7. $9.2455 - 10$.
8. $4.1335 - 10$.
9. $5,7875 - 6$.
10. $\overline{4}.1335$.
11. 6.9154.
12. 8.3464.
13. 0.8882.
14. $\overline{5}.9689$.

If the mantissa of a given logarithm, as 1.5271, is not in the table, the antilogarithm is obtained by interpolation as follows:

The mantissa 5271 lies between

.5263, the mantissa of the sequence 336,
and .5276, the mantissa of the sequence 337.

Therefore the antilogarithm of 1.5271 lies between 33.6 and 33.7. Since the tabular difference is 13 and the difference between .5263 and .5271 is 8, the mantissa .5271 lies $\frac{8}{13}$ of the way from .5263 to .5276. Therefore the required antilogarithm lies $\frac{8}{13}$ of the way from 33.6 to 33.7.

Then $\text{antilog}\,1.5271 = 33.6 + \frac{8}{13} \times .1,$

and $33.6 + .061 = 33.66.$

Therefore when the mantissa is not found in the table we have the

Rule. *Write the number of three figures corresponding to the lesser of two mantissas between which the given mantissa lies.*

Subtract the less mantissa from the given one and divide the remainder by the tabular difference to two decimal places. If the second digit is 5 or more, increase the first digit by 1; if less than 5, omit it.

Annex the resulting digit to the three already found and place the decimal point where indicated by the characteristic.

EXERCISES

Find the antilogarithms of the following:

1. 1.5523.
2. 2.3821.
3. 0.6790.
4. 2.5720.
5. $\bar{1}.2566$.
6. 7.3572 – 10.
7. 9.8327 – 10.
8. 5.9613 – 8.
9. 9.2664 – 10.
10. .7729.
11. 7.1060 – 10.
12. 6.2318 – 10.

245. Multiplication. Multiplication by logarithms depends on the

Theorem. *The logarithm of the product of two numbers is the sum of the logarithms of the numbers.*

That is, for the numbers a and x

$$\log_b (a \cdot x) = \log_b a + \log_b x.$$

Proof. Let $\log_b a = l_1$, (1)

and $\log_b x = l_2$. (2)

From (1), $a = b^{l_1}$. (3)

From (2), $x = b^{l_2}$. (4)

(3) × (4), $ax = b^{l_1 + l_2}$. (5)

Therefore
$$\begin{aligned} \log_b ax &= l_1 + l_2 \\ &= \log_b a + \log_b x. \end{aligned}$$

EXERCISES

Perform the indicated operation by logarithms:

1. 18 × 25.

Solution.
$$\begin{array}{r} \log 18 = 1.2553 \\ \log 25 = 1.3979 \\ \hline \log (18 \times 25) = 2.6532 \end{array} \text{ (adding)}$$
$$\text{antilog } 2.6532 = 450.$$

2. 37 × 28.
3. 29 × 9.
4. 9.8 × 6.
5. 42 × 3.3.
6. 386 × 27.
7. 432 × 263.
8. 589 × 375.
9. 4326 × 497.
10. 2870 × 3754.
11. 286.7 × 2.391.
12. 3.412 × 2.526.
13. 432 × .574.

Solution.
$$\begin{array}{rcl} \log 432 = 2.6355 = & 2.6355 & \\ \log .574 = \bar{1}.7589 = & 9.7589 - 10 & \\ \hline \log (432 \times .574) = 2.3944 = & 12.3944 - 10 & \text{(adding)} \end{array}$$
$$\text{antilog } 2.3944 = 247.9.$$

Since the *mantissa is always positive,* any number carried over from the tenths' column to the units' column is positive. This occurs in the preceding solution, where $.6 + .7 = 1.3$, giving $+1$ to be added to the sum of the characteristics $+2$ and -1, in the units' column. Mistakes in such cases will be few if the logarithms with negative characteristics be written as in the $9-10$ notation on the right.

In the preceding example and in others which follow, two methods are given for writing the logarithms which have negative characteristics. This is done to illustrate those cases in which the second of the two ways is preferable. It should be understood that in practice one, but not necessarily both, of these methods is to be used.

14. $385 \times .647.$

15. $571 \times .073.$

16. $37.6 \times .00865.$

17. $.0476 \times 673.$

18. $.07325 \times 6.354.$

19. $.6381 \times -.01897.$

HINT. Determine by inspection the sign of the product. Then operate as if all signs were positive.

20. $675 \times -.0286.$

21. $-.437 \times -.0046.$

246. Division. Division by logarithms depends on the

Theorem. *The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

That is, for the numbers a and x

$$\log_b \frac{a}{x} = \log_b a - \log_b x.$$

Proof. Let $\log_b a = l_1,$ (1)

and $\log_b x = l_2.$ (2)

From (1), $a = b^{l_1}.$ (3)

From (2), $x = b^{l_2}.$ (4)

(3) ÷ (4), $\frac{a}{x} = b^{l_1 - l_2}.$

Therefore

$$\log_b \frac{a}{x} = l_1 - l_2$$
$$= \log_b a - \log_b x.$$

EXERCISES

Divide, using logarithms:

1. $891 \div 27.$

Solution.

$$\begin{array}{r}
\log 891 = 2.9499 \\
\log 27 = 1.4314 \\
\hline
\log(891 \div 27) = 1.5185 \text{ (subtracting)} \\
\text{antilog } 1.5185 = 33.
\end{array}$$

2. $96 \div 32.$

3. $888 \div 47.$

4. $976 \div 361.$

5. $439 \div 27.1.$

6. $3860 \div 4.32.$

7. $4627 \div 281.$

8. $9896 \div 52.78.$

9. $6732 \div 7.81.$

10. $3.26 \div .0482.$

Solution.

$$\begin{array}{l}
\log 3.26 = 0.5132 = 10.5132 - 10 \\
\log .0482 = \bar{2}.6830 = 8.6830 - 10 \\
\hline
\log(3.26 \div .0482) = 1.8302 = 1.8302 - 0 \\
\text{antilog } 1.8302 = 67.64.
\end{array}$$

11. $2.35 \div .0683.$

12. $4.86 \div .751.$

13. $.0635 \div .277.$

14. $.2674 \div 3.66.$

15. $.07882 \div 68.72.$

16. $356 \times 392 \div 128.$

17. $\dfrac{347 \times (-625)}{346}.$

18. $\dfrac{473.2 \times 4.78}{-68.3}.$

19. $\dfrac{9.63 \times .0892}{.00635}.$

247. Involution. Involution by logarithms depends on the

Theorem. *The logarithm of the rth power of a number is r times the logarithm of the number.*

That is, for the numbers r and x, $\log_b x^r = r \log_b x$.

Proof. Let $\log_b x = l.$ (1)

Then $x = b^l.$ (2)

Raising both members of (2) to the rth power,

$$x^r = b^{rl}.$$

Therefore

$$\begin{aligned}\log_b x^r &= rl \\ &= r \log_b x.\end{aligned}$$

EXERCISES

Compute, using logarithms:

1. $(2.73)^3$.

Solution.

$$\log 2.73 = .4362.$$

$$\log (2.73)^3 = 1.3086 \text{ (multiplying by 3).}$$

$$\text{antilog } 1.3086 = 20.33.$$

2. $(6.32)^4$. **3.** $(34.26)^2$. **4.** $(6.715)^3$.

5. $(.425)^3$.

Solution.

$$\log .425 = \bar{1}.6284 = 9.6284 - 10.$$

$$\log (.425)^3 = \bar{2}.8852 = 28.8852 - 30.$$

$$\text{antilog } \bar{2}.8852 = .07677.$$

6. $(.362)^4$.

7. $(.0972)^2$.

8. $(.003597)^5$.

9. $(486.2)^2 \cdot (3.85)^3$.

10. $(.375)^5 \cdot (62.5)^4$.

11. $(2.25)^4 + (1.232)^3$.

248. Evolution. Evolution by means of logarithms depends on the

Theorem. *The logarithm of the real rth root of a number is the logarithm of the number divided by r.*

That is, for the real numbers r and n, $\log_b \sqrt[r]{n} = \frac{1}{r} \log_b n$.

Proof. Let $\log_b n = l$. (1)

Then $n = b^l$. (2)

Extracting the rth root of both members of (2),

$$(n)^{\frac{1}{r}} = (b^l)^{\frac{1}{r}} = b^{\frac{l}{r}}. \tag{3}$$

Therefore

$$\log_b (n)^{\frac{1}{r}} = \frac{l}{r} = \frac{\log_b n}{r}. \tag{4}$$

N	0	1	2	3	4	5	6	7	8	9	D
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8

N	0	1	2	3	4	5	6	7	8	9	D
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

EXERCISES

Compute, using logarithms:

1. $\sqrt[3]{376}$.

Solution. $\log 376 = 2.5752.$

$\log \sqrt[3]{376} = .8584$ (dividing by 3).

Then antilog $.8584 = 7.218.$

2. $\sqrt[3]{783}$. **3.** $\sqrt[5]{1435}$. **4.** $\sqrt[4]{3421}$.

5. $\sqrt[3]{.000639}$.

Solution. $\log .000639 = \bar{4}.8055.$

If one divided $\bar{4}.8055$ as it stands by 3, he would be likely to confuse the negative characteristic and the positive mantissa. This and other difficulties may easily be avoided by adding to the characteristic and subtracting from the resulting logarithm any integral multiple of the index of the root which will make the characteristic positive.

Thus $\log .000639 = 2.8055 - 6.$

$\log \sqrt[3]{.000639} = .9352 - 2$ (dividing by 3).

Then antilog $\bar{2}.9352 = .08614.$

6. $\sqrt{.0756}$.

7. $\sqrt[5]{.0007624}$.

8. $\sqrt[4]{.005679}$.

9. $(38.4)^{\frac{2}{3}}$.

10. $(4.925)^{\frac{3}{4}}$.

11. $(-6.387)^{\frac{5}{3}}$.

12. $-\sqrt{\dfrac{283 \cdot 7.627}{(3.423)^3}}$.

13. $\sqrt{\dfrac{(43.56)^2 \cdot 7.984}{(7.623)^3}}$.

14. $\sqrt[11]{269}$.

15. $\sqrt{87\sqrt[4]{463}}$.

16. $\frac{25}{381}\sqrt{\frac{137}{67}\sqrt[3]{972}}$.

17. $\frac{31}{426}\sqrt[3]{.07241}$.

18. Determine the logarithms of 5732, 573.2, 57.32, and 5.732 to the base 10 and to the base 100. Compare the results. What fact about logarithms do these results emphasize?

Note. The preceding four-place table will usually give results correct to one half of one per cent. Five-place tables give the mantissa to five decimal places of the numbers from 1 to 9999 and, by interpolation, the mantissa of numbers from 1 to 99,999. Such tables give results correct to one twentieth of one per cent, a degree of accuracy which is sufficient for most engineering work.

Six-place tables give the mantissa to six decimals for the same range of numbers as a five-place table, but the labor of using a six-place table is much greater than that of using a five-place one.

Seven-place tables contain the mantissas of the numbers from 1 to 99,999. Such tables are needed in certain kinds of engineering work and are of constant use in astronomy.

In place of a table of logarithms engineers often use an instrument called a slide rule. This is really a mechanical table of logarithms arranged ingeniously for rapid practical use. Results can be obtained with such an instrument far more quickly than with an ordinary table of logarithms, and that without recording or even thinking of a single logarithm. A slide rule ten inches long usually gives results correct to three figures. In work requiring greater accuracy a larger and more elaborate instrument which gives a five-figure accuracy is used.

249. Exponential equations. An **exponential equation** is an equation in which the unknown occurs in an exponent.

Many exponential equations are readily solved by means of logarithms, since $\log a^x = x \log a$.

Thus let $a^x = c$. Then $x \log a = \log c$. Whence $x = \log c \div \log a$.

EXAMPLE

Solve for x $\quad 8^x = 324.$

Solution. $\quad \log 8^x = \log 324,$

or $\quad x \log 8 = \log 324.$

Whence $$x = \frac{\log 324}{\log 8} = \frac{2.5105}{.9031} = 2.78+.$$

The student must overcome his hesitation actually to divide one logarithm by another if, as here, it is necessary.

MISCELLANEOUS EXERCISES

1. Can you find the logarithm of a negative number to a positive base? Explain.

Find, without reference to the table, the numerical values of:

2. $\log_3 9$.

3. $\log_2 8$.

4. $\log_8 2$.

5. $4 \log_9 27$.

6. $5 \log_{27} 9$.

7. $\log_4 8 + 3 \log_8 4$.

8. $2 \log_{27} 81 - 4 \log_{81} 27$.

9. $3 \log_{25} 125 + 2 \log_5 25 - 2 \log_{125} 5$.

10. $4 \log_3 (\frac{1}{3}) - 5 \log_9 (\frac{1}{27}) + 2 \log_{27} 9$.

Simplify:

11. $\log \frac{5}{8} + \log \frac{24}{25}$.

12. $\log \frac{7}{32} - \log \frac{35}{64}$.

13. $\log \frac{25}{4} + \log \frac{36}{50} - \log \frac{3}{4}$.

14. $2 \log 3 + 3 \log 2$.

15. $3 \log 4 + 4 \log 3 - 2 \log 6$.

Show that:

16. $\log \left(a - \frac{x^2}{a}\right) = \log (a + x) + \log (a - x) - \log a$.

17. $\log \sqrt{a^2 - x^2} = \frac{1}{2} [\log (a + x) + \log (a - x)]$.

18. $\log \sqrt{s(s - a)} = \frac{1}{2} [\log s + \log (s - a)]$.

19. $\log \sqrt{\frac{s(s - b)(s - c)}{s - a}} =$
$\frac{1}{2} [\log s + \log (s - b) + \log (s - c) - \log (s - a)]$.

Solve, using logarithms (obtain results to four figures):

20. The circumference of a circle is $2 \pi R$. ($\pi = 3.1416$, R = radius.)

(a) Find the circumference of a circle whose radius is 85 inches.

(b) Find the radius of a circle whose circumference is 3281 centimeters.

21. The area of a circle is πR^2.

(*a*) Find the area of a circle whose radius is 5.672 feet.

(*b*) Find the radius of a circle whose area is 67.37 square feet.

22. The area of the surface of a sphere is $4\,\pi R^2$.

(*a*) The radius of the earth is 3958.79 miles. Find its surface.

(*b*) Find the length of the equator.

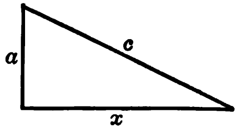
23. The volume of a sphere is $\frac{4\,\pi R^3}{3}$.

(*a*) Find the radius of a sphere whose volume is 86 cubic feet.

(*b*) Find the diameter of a sphere whose volume is 47 cubic inches.

24. If the hypotenuse and one leg of a right triangle are given, the other leg can always be computed by logarithms.

In the adjacent figure let a and c be given and x required.



Then

$$x = \sqrt{c^2 - a^2} = \sqrt{(c + a)(c - a)}.$$

Whence

$$\log x = \tfrac{1}{2}\log(c + a) + \tfrac{1}{2}\log(c - a).$$

(*a*) The hypotenuse of a right triangle is 377 and one leg is 288. Find the other leg.

(*b*) The hypotenuse of a right triangle is 1285 and one leg is 924. Find the other leg.

25. The area of an equilateral triangle whose side is s is $\frac{s^2}{4}\sqrt{3}$. Find in square feet the area of an equilateral triangle whose side is 34.23 inches.

Solve for x:

26. $3^x = 25.$

27. $64^x = 4.$

28. $16^x = 1024.$

29. $(-2)^x = 64.$

30. $3 = (1.04)^x.$

31. $2^x = 64.$

32. $4^{2x+1} = 84.$

33. $3^{x+3} = 6561.$

34. $10^{x-1} = 3.$

35. $8^{x+2} = 6.$

36. $(.3)^{-x} = 5.$

37. $(.07)^x = 9.$

Find the number of digits in:

38. (*a*) 3^{52}; (*b*) 2^{640}; (*c*) $2^9 \cdot 3^8 \cdot 5^7$.

39. In how many years will \$1 double itself at 3% interest compounded annually ? [*]

Solution. At the end of one year the amount of \$1 at 3% is \$1.03; at the end of two years it is \$(1.03) (1.03), or $\$(1.03)^2$; at the end of three years it is $\$(1.03)^3$, and at the end of x years it is $\$(1.03)^x$.

If x is the number of years required, $(1.03)^x = 2$.

Taking the logarithms of both members of the equation,

$$x \log 1.03 = \log 2.$$

Solving, $$x = \frac{\log 2}{\log 1.03} = \frac{.3010}{.0128} = 23.5+.$$

40. In how many years will \$1 double itself at 6% interest compounded annually ?

41. In how many years will any sum of money treble itself at 4% interest compounded annually ?

42. In how many years will \$450 double itself at $3\frac{1}{2}\%$ interest compounded annually ?

43. In how many years will \$4000 amount to \$7360.80 at 5% interest compounded annually ?

44. About 300 years ago the Dutch paid \$24 for the island of Manhattan. At 4% compound interest, what would this payment amount to at the present time ?

45. In how many years will \$12 double itself at 3% interest compounded semiannually ?

46. Show that the amount of P dollars in t years at $r\%$ interest compounded annually is $P(1+r)^t$; compounded semiannually is $P\left(1+\frac{r}{2}\right)^{2t}$; compounded quarterly is $P\left(1+\frac{r}{4}\right)^{4t}$; and compounded monthly is $P\left(1+\frac{r}{12}\right)^{12t}$.

[*] In making computations of this nature by the aid of logarithms, care must be exercised not to retain more significant figures in the result than are given with accuracy by the process.

47. Find the amount of $5000 at the end of four years, interest at 4% compounded (*a*) annually; (*b*) semiannually; (*c*) quarterly.

48. Find the amount of $4.12 at the end of five years, interest at 4%, compounded quarterly.

49. Set up and solve the equation used to determine the amount which should be paid for a $5 certificate to be paid in five years, interest at 4%, compounded quarterly.

NOTE. It is not a little remarkable that just at the time when Galileo and Kepler were turning their attention to the laborious computation of the orbits of planets, Napier should be devising a method which simplifies these processes. It was said a hundred years ago, before astronomical computations became so complex as they now are, that the invention of logarithms, by shortening the labors, doubled the effective life of the astronomer. To-day the remark is well inside the truth.

In the presentation of the subject in modern textbooks a logarithm is defined as an exponent. But it was not from this point of view that they were first considered by Napier. In fact it was not till long after his time that the theory of exponents was understood clearly enough to admit of such application. This relation was noticed by the mathematician Euler, about one hundred and fifty years after logarithms were invented.

It was by a comparison of the terms of certain arithmetical and geometrical progressions that Napier derived his logarithms. They were not exactly like those used commonly to-day, for the base which Napier used was not 10. Soon after the publication (1614) of Napier's work, Henry Briggs, an English professor, was so much impressed with its importance that he journeyed to Scotland to confer with Napier about the discovery. It is probable that they both saw the necessity of constructing a table for the base 10, and to this enormous task Briggs applied himself. With the exception of one gap, which was filled in by another computer, Briggs's tables form the basis for all the common logarithms which have appeared from that day to this.

The square roots and the cube roots on the following page are corrected to the nearest digit in the third decimal place.

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	52	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	55	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	57	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	58	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	60	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	65	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	68	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	69	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	70	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	71	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	73	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	77	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	78	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	81	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	84	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	86	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	88	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	89	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	90	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	92	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	95	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	97	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.642

1
25 √2752

1.5
225 √3375
225
1125
1125

216
− 67
8

INDEX

Abacus, 186
Abel, 452, 462
Absolute value, 22
Aggregation, signs of, 64
Ahmes, 442
Al-jebr, 57
Alternation, 466
Antecedent, 463
Antilogarithm, 488
Apothem, 356
Arabic notation, 1, 9, 186
Arabs, 5, 11, 57, 93, 186, 249, 254, 261
Archimedes, 274
Arrangement, 75
Associative Law of Multiplication, 70, 71
Axes, x- and y-, 202, 319
Axiom, 39, 40, 175

Babylonians, 157
Bacon, Roger, 186
Base, 480, 485
Binomial, 35, 82; square of, 105
Binomial theorem, 455; extraction of roots by, 458; rth term of $(a+b)^n$, 460–461
Binomials, difference of two squares, 124, 297; powers of, 455; product of, 107, 109, 110; sum or difference of two cubes, 133, 300
Braces, 64
Brackets, 64
Briggs, 481, 501

Cancellation, 149, 167
Cardan, 279
Cauchy, 452
Characteristic, 483
Check, 36, 42, 45, 46, 51
Circle, area, 15; circumference, 191
Coefficient, 10
Coefficients, literal, 95; polynomial, 38, 96
Commutative Law of Multiplication, 70
Complex number, 387
Conjugate imaginaries, 390
Consequent, 463
Constant, 472, 474, 476
Constant term, 137
Coördinates, 203
Cube root, 11, 114, 251
Cubes, of numbers, 502; sum or difference of, 133
Cubic equation, 141, 275
Cylinder, surface, 191; volume, 16

Decimal point, 186, 249
Decimals, 249; equations containing, 184; repeating, 343
Degree, 75; of an equation, 137, 141; of a term, 75, 76
Denominator, lowest common, 154
Diagonal of rectangle, 248
Diophantos, 278, 279
Discriminant of a quadratic equation, 400
Distance, x- and y-, 202
Division, by logarithms, 491; rule for, 88, 91

Egyptians, 157
Ellipse, 408
Equation, 6, 39, 54, 95; graph of, 205; indeterminate, 217; in one unknown, 55; in several unknowns, 217; in two variables, 202; of condition, 54; of fourth degree, 275; of second degree, 137; of third degree, 141; root of, 55, 218, 264; solution of, 41;

solving, 39; translation of problem into, 60, 82. *See also* Linear, Quadratic, Cubic
Equations, containing decimals, 184; containing fractions, 175, 180; incompatible, 221; indeterminate system of, 221; literal, 95, 190; literal, in two unknowns, 231; radical, 264. *See also* Systems, Simultaneous systems
Euclid, 32, 157, 261, 334
Euler, 462, 501
Evolution, law of, 330; by logarithms, 493
Exponent, 9, 210; logarithm as an, 481; meaning of a fractional, 330; meaning of a negative, 331; meaning of a zero, 331
Exponents, fractional, 251; law of, in division, 87; law of, in multiplication, 71
Expressions, integral, 295; rational, 295; reducible to difference of two squares, 299
Extremes, 465

Factor, 9, 10; common monomial, 116; highest common, 308; rationalizing, 266, 352; zero, 138
Factor Theorem, 301
Factorial notation, 460
Factoring, 113 ff.; definition of the process, 295; general directions for, 135, 305; solution of equations by, 137, 307
Factors, prime, 112, 295
Falling body, 17, 214
Fractions, 1; addition of, 157, 311; algebraic, 148; changes of sign in, 162, 163; complex, 172; division of, 170; equations containing, 175, 180; equivalent, 154; history of, 157; lowest common denominator of, 154; lowest terms of, 149; multiplication of, 167; operations with, 157; reduction of mixed expressions to, 165; simultaneous systems containing, 224; subtraction of, 157; terms of, 148
Function, definition of a, 359; graph of a cubic, 362; graph of a linear, 360; graph of a quadratic, 361; notations for a, 360
Functions, 359; names of, 359

Galileo, 501
Gauss, 364, 387, 393, 452
Graph, 200; of a cubic function, 362; of an equation, 205; of linear equation, 205, 206, 208; of a linear function, 360; of a quadratic equation in two variables, 405
Graphical representation, of linear system in three unknowns, 239; of statistics, 210, 412
Graphical solution of an equation in one unknown, 364; the process of, 365
Graphical solution of a quadratic system in two variables, 409
Greeks, 93, 157

Harriot, 5
Highest common factor, 308
Hindus, 5, 21, 28, 29, 157, 186, 261, 279
Hyperbola, 406
Hypotenuse, 247

Identity, 54; sign of, 55
Imaginaries, addition and subtraction of, 387; conjugate, 390; definitions, 386; division of, 390; equations with imaginary roots, 392; factors involving, 394; multiplication of, 388; note on the use of, 394
Imaginary, 343
Imaginary roots, 366
Index, 11, 251
Infinite geometrical series, 448
Integer, 1, 250; consecutive, 44; consecutive even, 44; consecutive odd, 44; even, 44; odd, 44; positive, 27
Integral expressions, 295
Interest, 15, 186; simple, 187
Interpolation, 486
Inversion, 466
Involution, law of, 330; by logarithms, 462

Irrational number, 250
Italians, 5

Kepler, 408, 501
Klein, Professor Felix, 364
Kronecker, Leopold, 27

La Place, 452
Leibnitz, Gottfried Wilhelm, 465
Lever, 193
Line, straight, 360
Linear and quadratic equations, systems of, 279
Linear equation, 137; in two variables, 208, 217, 224
Linear systems in three unknowns, 236
Literal coefficients, 95; equations with, 95, 190, 231, 277
Logarithms, definition of, 480; antilogarithm, 488; interpolation, 486; systems of, 481; table of, 494–495
Lowest common denominator, 154
Lowest common multiple, 152
Lowest terms, 149

Mantissa, 483, 491
Means, 464; arithmetical, 435; geometrical, 445
Member, left, 39; right, 39
Monomial, 33; cube root of, 114; square root of, 113
Monomial denominator in equations, 175
Monomial factor, 116
Monomials, addition of, 33, 34; division of, 87; division of polynomials by, 89; multiplication of, 72; multiplication of polynomials by, 73; subtraction of, 49
Moors, 186
Morse, S.F.B., 275
Motion, uniform, 97, 192
Multiple, lowest common, 152
Multiplication, rules for, 71–74; rule for, by logarithms, 490

Napier, Sir John, 186, 462, 484, 501
Negative numbers, 19, 20, 210, 279. *See also* Positive and negative numbers
Newton, 367, 451, 462
Numbers, classification of, 343; complex, 387; imaginary, 250; irrational, 250; orthotomic, 386; pure imaginary, 386; rational, 250; real, 343
Numerals, 1, 9; arabic, 9
Numerical value, 15, 22

Operation, signs of, 1, 5, 30
Operations, order of, 13
Order, 251; in addition, 34
Ordinate, 202
Origin, 203
Oughtred, 5, 465

Parabola, 405
Parentheses, 10, 13, 30, 34, 64, 79, 82; insertion, 68; removal, 64
Pascal's triangle, 456
Percentage, 186
Points, plotting, 204
Polynomial, 35
Polynomial denominators in equations, 180
Polynomials, addition of, 35; division of, by monomials, 89; division of, 91; multiplication of, by monomials, 73; multiplication of, 74; prime to each other, 152; subtraction of, 50; with common monomial factor, 116
Positive and negative numbers, 20; addition of, 21; division of, 28; multiplication of, 26; subtraction of, 23
Power, 75
Powers, ascending, 75; descending, 75; sum or difference of two like, 303
Prime, 113, 152, 308
Probability curve, 415
Problems, solution of, 45
Product, 9, 10, 70, 71, 93, 107, 109, 110; of sum and difference, 107
Products, important special, 105, 287
Progressions, common difference of arithmetical, 432; definition

of arithmetical, 432; definition of geometrical, 442; *n*th term of arithmetical, 433; *n*th term of geometrical, 444; ratio of geometrical, 443

Proportional, fourth, 465; mean, 465; third, 465

Proportions, derived by addition, by subtraction, and by addition and subtraction, 467; from equal products, 465

Quadratic and linear equations, systems of, 279

Quadratic equation, 137, 275; character of the roots of, 400; comparison of the various methods of solution of the, 374; constant term of, 137; formation of, with given roots, 396; history of, 278; in two unknowns, 279; number of roots of, 402; pure, 272; relation between the roots and the coefficients of, 397; solution of, by completing the square, 270; solution of, by factoring, 137; solution of, by formula, 297, 371; with literal coefficients, 277

Quadratic expressions, factors of, 403

Quadratic trinomial, 127, 298; the general, 130

Quantity, 471

Radical, 250; sign, 11

Radical equations, 264

Radicals, addition of, 259; conjugate, 266; division of, 266; multiplication of, 261; simplification of, 253; subtraction of, 259

Radicand, 251

Radicands, fractional, 256

Raleigh, Sir Walter, 5

Ratio, 463

Rational expressions, 295

Rational number, 250

Rationalizing factor, 266, 352

Recorde, 5, 242

Remainder Theorem, 300

Roman notation, 186

Romans, 157

Root, cube, 11, 114; of an equation, 55, 217, 264; principal, 251; principal square, 114; square, 11, 113

Roots, cube, 251; even, 114; fourth, 114; imaginary, 366; imaginary, for a cubic equation, 367; of a monomial, 113, 114, 115; rejected, 143; set of, 218; sixth, 114; table of square and cube, 502

Series, arithmetical, 438; geometrical, 447; infinite geometrical, 448; sum of, 438

Set of roots, 218

Sign, changes of, in fraction, 162, 163; double, 114; of equality, 2, 5; of identity, 55; radical, 11

Simultaneous systems, 218; containing fractions, 224; solution of, by addition and subtraction, 219; solution of, by substitution, 222

Slide rule, 497

Solution, definition of, 218; of equations, 15, 41; of equations by factoring, 137 ff.; graphical, of linear equations in two variables, 208; of problems, 6, 45; of systems, 219, 222

Solving an equation, 39

Square, of binomial, 105; of trinomial, 288; perfect, 121; trinomial, 121

Square root, 11; historical note on, 249; of algebraic expressions, 240; of numbers, 243; principal, 114

Squares, difference of, 124, 297

Squares of numbers, 502

Stifel, 11, 279, 458

Subscripts, 191

Substitution, solution of systems by, 222

Sum, algebraic, 23

Surd, 344

System, indeterminate, 221; in three unknowns, 236; of equations, 217; simultaneous, 218

Systems, containing fractions, 224; determinate, in three unknowns, 325; equivalent, 424; incompatible or inconsistent, 221, 323; of linear and quadratic equations, 279; simultaneous, 323; solution by addition and subtraction, 219, 323; solution by substitution, 222, 322; solution of, when one equation is linear and the other quadratic, 279, 416; solution of, when both are quadratic, 419; solution of a linear, in two variables, by graphs, 208; special methods for solution of, 422; symmetric, 421

Table, of cubes and squares, 502; of logarithms, 494–495
Term, 33
Terms, dissimilar, 34; of fraction, 148; similar, 33
Tartaglia, 456
Theon, 249
Transposition, 56
Trapezoid, 146; altitude of, 146; area of, 146
Triangle, altitude of, 145; area of, 15, 145, 191; equilateral, 248, 258; isosceles right, 258; right, 247, 258, 259
Trinomial, 35; general quadratic, 130; quadratic, 127; square of a, 288
Trinomial squares, 121

Unknown, 6, 39, 95

Variable, 217
Variation, 471; direct, 471; inverse, 473; joint, 475
Vinculum, 64

Wallis, John, 465

Zero, as denominator, 161; as an exponent, meaning of, 331; as factor, 138; division by, 29, 40; multiplication by, 29

Solve $\sqrt{x+15} - \sqrt{x+3} = 2\sqrt{x}$

" $\sqrt{cx+ab} + \sqrt{cx-ab} = \sqrt{4cx-a^2}$

Mult $5\sqrt{3} + 3\sqrt{5} + 4\sqrt{7}$ by $5\sqrt{3}$
$3\sqrt{5} - 4\sqrt{7}$

Mult $\sqrt{a+b} + \sqrt{a-b}$ by $\sqrt{a+b} + \sqrt{a-b}$

Divide $\sqrt{x} - \sqrt{y}$ by $\sqrt{x} + \sqrt{y}$

$\sqrt{x-2} + 1$ by $\sqrt{x-2} + 2$

Find to 3 dec $\frac{6}{3+\sqrt{15}}$ Series [illegible]

Solve by completing the sq
$6x^2 + 14 = 41x$

Solve

$x^2 + ax$ [illegible]

$$\begin{cases} x^2 + y^2 = 101 \\ x^2 + xy = 7 \end{cases}$$

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